Scuola Internazionale Superiore di Studi Avanzati

PhD Thesis

Symplectic Topology, Mirror Symmetry and Integrable systems

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Contents

Introduction

Since the fundamental work of Witten and Kontsevich ([23],[36]), Gromov's powerful tool of holomorphic curves and the subsequent successful Gromov-Witten theory showed a remarkable (and quite involved) link between the topology of symplectic manifolds and integrable systems of Hamiltonian PDEs. This phenomenon, studied and encoded into the geometric structure of Frobenius manifolds by Dubrovin (see e.g. [13]), provided a bridge to fruitfully exchange insight and results between the two disciplines.

Thanks to the program of Symplectic Field Theory (SFT) initiated by Eliashberg, Givental and Hofer ([14],[16]), a new, very suggestive and more direct link has been discovered in the study of holomorphic curves in directed symplectic cobordisms between contact manifolds (or, more in general, manifolds with a stable Hamiltonian structure). The hope is that this theory actually helps in explaining and understanding the above relation. In Symplectic Field Theory the integrable structure encodes some geometric properties at the level of top dimensional stratum of the boundary of the relevant moduli space of curves. This stratum (as in Floer theory) has codimension one, so the Gromov-Witten-like invariants which one considers are directly sensitive to its effects. In the ordinary Gromov-Witten case, on the contrary, these phenomena are hidden in higher codimension, and their effects are much more subtle.

More explicitly, Symplectic Field Theory [16] is a fairly new branch of symplectic topology studying holomorphic curves in symplectic manifolds with ends in the spirit of Gromov-Witten theory (which it actually contains as a special case). The presence of cylindrical ends that are symplectizations of contact manifolds and, in particular, the imposition (boundary condition) that holomorphic curves have punctures that asymptotically coincide with Reeb orbits in these ends, gives to the theory a rich algebraic structure where classical and quantum integrable systems, together with many natural tools of that context, arise.

In particular, let W be a symplectic manifold with cylindrical ends, i.e. a symplectic cobordism between contact manifolds V^+, V^-, completed by attaching to these boundaries their symplectizations $V^+ \times [0, +\infty)$, $V^- \times (-\infty, 0]$. The potential counting holomorphic curves in each of the two symplectizations $V^+ \times \mathbb{R}$, $V^- \times \mathbb{R}$, asymptotically cylindrical over Reeb orbits of V^+, V^-, and intersecting at marked points some given homology cycles, is to be interpreted as the Hamiltonian for a (quantum) Hamiltonian system. Schrödinger equation (and its semiclassical limit, Hamilton-Jacobi equation) relative to this Hamiltonian enters in the computation of the Gromov-Witten potential of W that, in this context, plays the role of the phase of the wave function, as explained in [16], section 2.7. Namely, this Hamiltonian evoulution encodes the dependence of the number of holomorphic curves in W

on the intersection conditions, at marked points, with homology cycles dual to compactly supported cohomology classes (recall that W is non-compact precisely along the $\mathbb{R}$-direction transversal to V^+ and V^-). Moreover similar formulas (see [16], section 2.5) can be used to deduce potentials for a composition of cobordisms from the ones of the single pieces.

It is fairly natural to wonder whether the (quantum) Hamiltonian systems arising this way in Symplectic Field Theory are in general integrable systems (it is so for all explicitly known cases, which are indeed quite few) and, if not, to investigate the topological properties of the target space which ensure completeness of such systems. Even more, the other Hamiltonians forming all the symmetries of the systems above, and their correponding Schrödinger equations, can be used in a way totally analogous to the one explained above to compute the descendant potential of W, whence the importance of understanding this rich structure.

This is a general idea whose details are far from being completely understood. However there are some geometric situations where the common geometric origin of integrable systems from Gromov-Witten and Symplectic Field Theory is quite evident. In particular, in the case of a prequantization bundle V over an integral symplectic manifold M (or, more in general, a Hamiltonian structure of fibration type), as showed in [16] and [4], we have an (at least at genus 0) explicit correspondence between the integrable hierarchies associated to V (via SFT) and M (via the Frobenius structure of quantum cohomology).

The general aim of this thesis is twofold:

- we investigate the role of (quantum) Hamiltonian, possibly integrable systems in Symplectic Field Theory. In particular we review the meaning and consequence of their appearence and identify a broad class of examples, namely framed Hamiltonian strucures of fibration type, where integrability is achieved and comes from the quantum cohomology structure of the base. In particular, with the advent of orbifold Gromov-Witten theory ([9],[1],[2]), this class can be extended to involve fibrations with singular fibers, hence with orbifold base. Moreover we examine how the geometry of Gromov-Witten gravitational descendents and their SFT analogue is linked to the symmetries of the given Hamiltonian system, related to the SFT potential with no psi-classes, while the higher genus expansion is related to the quantization of the $g = 0$ classical Hamiltonian system.
- we use Symplectic Field Theory as a mean to costruct Frobenius manifolds and, hence, integrable systems of PDEs. This is done basically by computing Gromov-Witten invariants of a closed symplectic manifold M, which give, the usual way, a Frobenius manifold structure on the cohomology of M. In fact Symplectic Field Theory, similarly to relative Gromov-Witten theory, provides powerful tools for determining the quantum cohomology of closed symplectic manifolds. Namely the above mentioned gluing formulas for composition of symplectic cobordisms can be applied to the computation of Gromov-Witten potentials by cutting the target closed manifold M along some closed contact hypersurface V. Each half M_1 and M_2, with $\partial M_1 = \partial M_2 = V$, then becomes a symplectic cobordism between the mentioned contact manifold V and the empty set. By properly choosing the contact hypersurface V, it is likely that computing the SFT potential of M_1 and M_2 is sensibly easier than directly attacking the quantum cohomology of M. This fact relies on reducing the cohomology of the target by this cutting

procedure and using Schrödinger equation to recover the dependence on the compactly supported cohomology classes, before gluing back.

The plan of the work is the following:

- In Chapter 1 we recall, basically from [16] and [14], the ideas and methods of Symplectic Field Theory. Our review will focus on the algebraic structure arising from topology, more than on the geometry underlying it. In particular we define the SFT analogue of the Gromov-Witten potential as an element in some graded Weyl algebra and consider its properties (grading, master equations, semiclassical limit). We then stress (following [18]) how this algebraic structure allows the appearence of a system of commuting differential operators (on the homology of the Weyl algebra) which can be thought of as a system of quantum Hamiltonian PDEs with symmetries. Sometimes this symmetries are many enough to give rise to a complete integrable system (at least at the semiclassical level) and we examine the main examples where this happens. Eventually we review some results of [16] which turn out to be very useful in computations and which we actually employ in the next chapters.

- In Chapter 2 we apply the methods of Symplectic Field Theory to the computation of the Gromov-Witten invariants of target Riemann surfaces. Our computations reproduce the results of [27], [28] which, in principle, solve the theory of target curves, but are fairly more explicit and, above all, clarify the role of the KdV hierarchy in this topological theory. More precisely we are able to describe the full descendant Gromov-Witten potential as the solution to Schrödinger equation for a quantum dispersionless KdV system. This quantization of KdV, already appearing in [31], can be easily dealt with in the fermionic formalism to give extremely explicit results, like closed formulae for the Gromov-Witten potential at all genera and given degree. These results where published by the author in [32].

- In Chapter 3 we use basically the same techniques of Chapter 2 to compute the Gromov-Witten theory of target curves with orbifold points (orbicurves). As in the smooth case, the coefficient for the Gromov-Witten potential are written in terms of Hurwitz numbers. It turns out that we can even classify those target orbicurves whose potential involves only a finite number of these a priori unknown Hurwitz coefficients, so that they can be determined using WDVV equations. These polynomial P^1-orbifolds are the object of our study for the final part of this work. Moreover, we extend the theorem by Bourgeois ([4]) about Hamiltonian structures of fibration type to allow singular fibers (Seifert fibrations), so that we can use our result on Gromov-Witten invariants of polynomial $\mathbb{P}^1$-orbifolds to deduce the SFT-Hamiltonians of the fibration.

- In Chapter 4 we completely solve the rational Gromov-Witten problem for polynomial P^1-orbifolds. Namely we find a Landau-Ginzburg model which is mirror symmetric to these spaces. This model consists in a Frobenius manifold structure on the space of what we call for brevity tri-polynomials, i.e. polynomials of three variables of the form $-xyz + P_1(x) + P_2(y) + P_3(z)$. The main results here are the explicit construction of the Frobenius manifold structure with closed expressions for flat coordinates and the mirror theorem 4.0.3, i.e. the isomorphism of this Frobenius structure with the one on the quantum cohomology of polynomial $\mathbb{P}^1$-orbifolds. From the polynomiality property of the Frobenius potentials involved, one is able to show that there is also a third mirror symmetric partner in the picture, namely the Frobenius manifold associated to extended affine Weyl groups of type A, D, E ([12]). The results of these last two chapters appeared in [33].

- In the Conclusions we summarize our results and analyze possible further developments and directions to be explored.

Chapter 1

Symplectic Field Theory and Integrable Systems

For the reader's convenience and also in order to fix notations, in this chapter we recall from [16] and [14] the main ideas of Symplectic Field Theory. In this brief review we wish to put a stress on the appearance of quantum Hamiltonian systems in this symplectic topological context. In this sense, part of the material was taken from [18], which is a simple and clear exposition.

What the reader will not find here is a careful exposition of the geometry of the moduli spaces of holomorphic curves with cylindrical ends which are involved in defining Symplectic Field Theory. Actually we will focus on the algebraic side of SFT and on how it is related with the symplectic topology of the target space. Indeed Symplectic Field Theory is still somewhat foundationally incomplete, since a rigorous transversality theorem for the Cauchy-Riemann operator is still missing (see [16], [5]). Such a result requires abstract perturbations of the Cauchy-Riemann equations via polyfold theory ([21]) and it is not the subject of our discussion.

Of course it is precisely the geometry of such moduli spaces, and in particular their compactification ([5]) and boundary strata, which is responsible of the interesting algebraic structure of the formalism. For instance the master equations (1.1) and (1.3) encode how the boundaries of one-dimensional moduli spaces are composed of pairs of holomorphic curves from zero dimensional moduli spaces, similarly to what happens with the *broken trajectories phenomenon* in Floer homology.

1.1 Symplectic preliminaries

A Hamiltonian structure (see also [14]) is a pair (V, Ω), where V is an oriented manifold of dimension $2n-1$ and Ω a closed 2-form of maximal rank $2n-2$. The line field $\mathrm{Ker}\Omega$ is called the characteristic line field and we will call characteristic any vector field which generates $\mathrm{Ker}\Omega$. A Hamiltonian structure is called stable if and only if there exists a 1-form λ and a characteristic vector field R (called Reeb vector field) such that

$$\lambda(R) = 1 \qquad \text{and} \qquad i_R \mathrm{d}\lambda = 0.$$

A framing of a stable Hamiltonian structure is a pair (λ, J) with λ as above and J an complex structure on the bundle $\xi = \{\lambda = 0\}$ compatible with Ω.

The two main examples of framed Hamiltonian structures we are going to consider arise from contact manifolds and S^1-bundles over symplectic manifolds. In the first case, given a contact manifold $(V, \xi = \{\alpha = 0\})$ with a compatible complex structure J on ξ we can consider the framed Hamiltonian structure $(V, \Omega = \mathrm{d}\alpha, \lambda = \alpha, J)$. In the second case, let (M, ω) be a symplectic manifold with a compatible almost complex structure J_M, $p : V \to M$ any S^1-bundle and λ any S^1-connection form over it; then $(V, \Omega = p^*\omega, \lambda, J)$, with J the lift of J_M to the horizontal distribution, is a framed Hamiltonian structure.

A symplectic cobordism $W = \overrightarrow{V^+V^-}$ between two Hamiltonian structures (V^+, Ω_+) and (V^-, Ω_-) is a symplectic manifold (W, Ω) such that $\partial W = V^+ \cup (-V^-)$ and $\Omega|_{V^\pm} = \Omega_\pm$. A framed symplectic cobordism between two framed Hamiltonian structures $(V^+, \Omega_+, \lambda_+, J_+)$ and $(V^-, \Omega_-, \lambda_-, J_-)$ is a symplectic cobordism (W, Ω) equipped with an almost complex structure J which is compatible with Ω and such that $J(\xi_\pm) = \xi_\pm$ where, as above, $\xi_\pm = \{\lambda_\pm = 0\}$.

1.2 Cylindrical cobordisms

A special example of framed symplectic cobordism is given by the cylinder $V \times \mathbb{R}$ over a framed Hamiltonian structure (V, Ω, λ, J). It is endowed with an almost complex structure, still denoted by J, which is invariant with respect to translations along $\mathbb{R}$, induces the given J on each slice $V \times \{t\}$ and satisfies $J(\frac{\partial}{\partial t}) = R$. In particular, the cylinder over a contact manifold (V, α) is its symplectization $(V \times \mathbb{R}, \mathrm{d}(\mathrm{e}^t \alpha))$. When not differently specified, in what follows we will consider the case of a generic Hamiltonian structure, for which the set of periodic Reeb orbits is discrete and finite.

We now introduce the algebraic formalism of Symplectic Field Theory. To each multiple γ of a periodic Reeb orbit in a framed Hamiltonian structure V of dimension $2n-1$, we assign two graded variables p_γ, q_γ with grading

$$\deg(p_\gamma) = -\mathrm{CZ}(\gamma) + (n-3)$$

$$\deg(q_\gamma) = +\mathrm{CZ}(\gamma) + (n-3)$$

We then choose a string $(\theta_1, \ldots, \theta_N)$ of differential forms on V and assign to each $\theta_i \in \Omega^*(V)$ a sequence of graded variables $t_{ij},\ j \in \mathbb{N}$ with grading

$$\deg(t_{ij}) = 2(j-1) + \deg(\theta_i)$$

Finally let $\hbar$ be another variable with degree $2(n-3)$.

To a framed Hamiltonian structure V we associate the graded Weyl algebra $\mathfrak{W}(V)$ (or simply $\mathfrak{W}$) over the group ring $\mathbb{C}(H_2(V))$ of power series in the variables $\hbar$, p_γ, t_{ij} with coefficients which are polynomials in the variables q_γ, equipped with the associative product where all the variables super-commute except for p_γ and q_γ corresponding to the same orbit γ, for which

$$[p_\gamma, q_\gamma] = \mu_\gamma \hbar$$

where μ_γ id the multiplicity of γ. The grading of the formal variable z for the group ring $\mathbb{C}(H_2(V))$ is given by

$$\deg(z^A) = -2c_1(A)$$

where $c_1(A)$ is the first Chern class of $\mathrm{T}V|_A$.

Now, to the cylinder $(V \times \mathbb{R}, J)$ over a framed Hamiltonian structure (V, Ω, λ, J), we associate the Hamiltonian (SFT-potential)

$$\mathbf{H} = \frac{1}{\hbar} \sum_{g=0}^{\infty} \mathbf{H}_g \hbar^g$$

defined by requiring that the coefficient of the monomial $\hbar^{g-1} t^I p^{\Gamma^+} q^{\Gamma^-}$, with $t^I = t_{i_1 j_1} \dots t_{i_r j_r}$, $p^{\Gamma^+} = p_{\gamma_1^+} \dots p_{\gamma_{s+}^+}$ and $q^{\Gamma^-} = q_{\gamma_1} \dots q_{\gamma_{s-}^-}$ is given by the integral

$$\int_{\mathcal{M}_{g,r,\Gamma^-,\Gamma^+}} \mathrm{ev}_1^* \theta_{i_1} \wedge \psi_1^{j_1} \wedge \dots \wedge \mathrm{ev}_r^* \theta_{i_r} \wedge \psi_r^{j_r}$$

where $\psi_i = c_1(L_i)$ is the i-th psi-class (like in Gromov-Witten theory) and the integral is over the moduli space of holomorphic curves in $V \times \mathbb{R}$ with r marked points and $s^\pm$ positive/negative punctures asymptotically cylindrical over the Reeb orbits $\Gamma^\pm$ and realizing, together with the chosen capping surfaces (see [16] for details), the homology cycle A in $V \times \mathbb{R}$, modulo the $\mathbb{R}$ action coming from the $\mathbb{R}$ symmetry of the cylindrical target space $V \times \mathbb{R}$.

The series $\mathbf{H}$ is an element of the graded Weyl algebra $\mathfrak{W}$. It is homogeneous of degree -1 and satisfies a structure equation in the form (see [16])

$$d\mathbf{H} + \frac{1}{2}[\mathbf{H}, \mathbf{H}] = 0$$

If the chosen string $\theta_1, \dots, \theta_N$ is formed only by closed differential forms, then the structure equation reduces to

$$[\mathbf{H}, \mathbf{H}] = 0 \tag{1.1}$$

1.3 Commuting operators on homology

Expanding the Hamiltonian $\mathbf{H}$ in powers of the formal variables t,

$$\mathbf{H} = \mathbf{H}^0 + \sum_{i,j} \mathbf{H}^1_{i,j} t_{ij} + \sum_{i_1,j_1,i_2,j_2} \mathbf{H}^2_{i_1 j_1; i_2 j_2} t_{i_1 j_1} t_{i_2 j_2} + \dots$$

defines sequences of elements $\mathbf{H}^0$, $\mathbf{H}^1_{i,j}$, $\mathbf{H}^2_{i_1 j_1; i_2 j_2}$ of elements in the subalgebra $\mathfrak{W}^0 \subset \mathfrak{W}$ where all t-variables are set to zero.

Moreover, expanding the same way the master equation $[\mathbf{H}, \mathbf{H}] = 0$ one gets that:

1) $D^0 = [\mathbf{H}^0, \cdot] : \mathfrak{W}^0 \to \mathfrak{W}^0$ makes $\mathfrak{W}^0$ into a differential algebra, since $D^0 \circ D^0 = 0$,

2) the commutator on $\mathfrak{W}^0$ descends to the homology $H_*(\mathfrak{W}^0, D^0)$, thanks to the Jacobi identity,

3) $[\mathbf{H}^1_{i_1 j_1}, \mathbf{H}^1_{i_2 j_2}] = 0$ as homology classes in $H_*(\mathfrak{W}^0, D^0)$.

It can, moreover, be proven ([16],[18]) that both the Weyl homology algebra $H_*(\mathfrak{W}^0, D^0)$ and the sequence of elements $\mathbf{H}_{ij}$ are actually symplectic invariants (independent from choices like almost complex structure, contact form, etc.).

Notice that, by taking the limit $\hbar \to 0$, the graded Weyl algebra $\mathfrak{W}$ transforms to a graded Poisson algebra $\mathfrak{P}$, and all the statements above can be rephrased in this limit. In particular one gets a system of Poisson commuting Hamiltonians $\mathbf{h}_{ij}$ over the homology Poisson algebra $H_*(\mathfrak{P}^0, d^0)$, where $d^0 = \{\mathbf{h}^0, \cdot\}$.

One can venture to interpret the Poisson algebra $\mathfrak{P}$ as the space of functions over a (infinite dimensional) phase space of vector valued formal Fourier series $\sum_\gamma p_k \mathrm{e}^{\mathrm{ikx}} + q_k \mathrm{e}^{-ikx}$, where $p_k = (p_{k\gamma_1}, \dots, p_{k\gamma_l})$ and $q_k = (q_{k\gamma_1}, \dots, q_{k\gamma_l})$ are canonical coordinates and $\{\gamma_1, \dots, \gamma_l\}$ is the set of periodic Reeb orbits of V. This way $H_*(\mathfrak{P}^0, d^0)$ can be seen as the space of functions of some reduction of the above phase space and the Hamiltonians $\mathbf{H}_{ij}$ define commuting Hamiltonian vector fields on it. The system of commuting operators $\mathbf{H}_{ij}$ in the full Weyl algebra $H_*(\mathfrak{W}^0, D^0)$ can then be regarded to as a quantization of the classical infinite-dimensional systems above.

In a number of nontrivial examples this set of compatible Hamiltonian flows achieves completeness in the sense of the modern theory of integrable systems of Hamiltonian PDEs (see [13]). In fact, all the computed examples can be grouped in the class of framed Hamiltonian structures of fibration type. For this situation there exist an explicit link between the integrable system associates to the Frobenius structure on quantum cohomology of the symplectic base M and the Symplectic Field Theory Hamiltonians associated to the framed Hamiltonian structure V of fibration type over M. The link is given by the following result, due to Bourgeois ([4],[16]).

We will need the non-generic Morse-Bott version of Symplectic Field Theory. Referring to [4] for the general construction, here we just stick to the case of fibrations (see above), where the space $\mathcal{P}$ of periodic Reeb orbits can be presented as $\mathcal{P} = \coprod_{k=1}^\infty \mathcal{P}_k$, where each $\mathcal{P}_k$ is a copy of the base manifold M. Let then $\Delta_1, \dots, \Delta_b$ be a basis of $H^*(M)$ such that the system of forms $\tilde{\Delta}_j := \pi^*(\Delta_j)$, $j = 1, \dots, c < b$ generate $\pi^*(H^*(M)) \subset H^*(V)$, and the forms $\tilde{\Theta}_1, \dots, \tilde{\Theta}_d$ complete it to a basis of $H^*(V)$. Suppose $H_1(M) = 0$ and choose a basis $A_0, A_1, \dots, A_N$ of $H_2(M)$ in such a way that $\langle c_1(V), A_0 \rangle = l$ (if l is the greatest divisor of the first Chern class $c_1(V)$ of our fibration), $\langle c_1(V), A_i \rangle = 0,\ i \neq 0$, and a basis of $H_2(V)$ is given by the lifts of $A_1, \dots, A_N$ if $l \neq 0$, $A_0, A_1, \dots, A_N$ if $l = 0$.

The graded Poisson algebra $\mathfrak{P}$, in this case, is formed by series in the variables $(t_{ij}; \tau_{i,j})$ associated to the string $(\tilde{\Delta}_1, \dots, \tilde{\Delta}_c; \tilde{\Theta}_1, \dots, \tilde{\Theta}_d)$, the variables $p_{k,1}, \dots, p_{k,b}$ associated to the the classes $(\Delta_1, \dots, \Delta_b)$ (instead of the periodic orbits themselves) and $\hbar$, with coefficients which are polynomials in the variables $q_{k,1}, \dots, q_{k,b}$. The Poisson structure is given by the

Poincaré pairing in $H^*(M)$. The grading of the variables is given by:

$$\deg(t_{ij}) = 2(j-1) + \deg(\tilde{\Delta}_i)$$
$$\deg(\tau_{ij}) = 2(j-1) + \deg(\tilde{Theta}_i)$$
$$\deg(q_{k,i}) = \deg(\Delta_i) - 2 + 2ck$$
$$\deg(q_{k,i}) = \deg(\Delta_i) - 2 - 2ck$$
$$\deg(z_i) = 2c_1(A_i)$$

where $c = \frac{\langle c_1(TM), A_0 \rangle}{l}$ (see [16] for details on how to deal with fractional degrees).

The Hamiltonian $\mathbf{h}$ is defined the usual way, but this time one has to pull-back the classes $\Delta_1, \ldots, \Delta_b \in H^*(M)$ through the evaluation map from the moduli space to the space of periodic Reeb orbits and integrate them too. The main result is the following

Proposition 1.3.1 ([4],[16]). *Let $\mathbf{f}_M(\sum t_{ij}\Delta_i, z)$ be the genus 0 orbifold Gromov-Witten potential of P and $\mathbf{h}_V(\sum t_{ij}\tilde{\Delta}_i + \sum \tau_{kl}\tilde{\Theta}_k, q, p)$ the rational SFT potential of V (as a framed Hamiltonian structure of fibration type). Let*

$$\mathbf{h}_{kl}(t,q,p) = \frac{\partial \mathbf{h}_V}{\partial \tau_{kl}} \left(\sum_1^c t_{ij}\tilde{\Delta}_i + \tau_{kl}\tilde{\Theta}_k, q, p \right) \Bigg|_{\tau_{kl}=0}$$
$$h_{kl}(t;z) = \frac{\partial \mathbf{f}_P}{\partial s_{kl}} \left(\sum_1^b t_{i0}\Delta_i + s_{kl}\pi_*\tilde{\Theta}_k, z \right) \Bigg|_{s=0}$$

where π_* denotes integration along the fibers of V. Then we have

$$\mathbf{h}_{kl}(t,q,p) = \frac{1}{2\pi}\int_0^{2\pi} h_{kl}(t_1 + u_1(x), \ldots, t_b + u_b(x), u_{b+1}(x), \ldots, u_c(x); \tilde{z})\mathrm{d}x$$

where

$$u_n(x) := \sum_{k=1}^{\infty} \left(q_{k,n}\, e^{-\mathrm{i}\,kx} + p_{k,n}\, e^{\mathrm{i}\,kx} \right) \qquad n = 1, \ldots, b$$

and $\tilde{z} = (\mathrm{e}^{-\mathrm{i}lx}, z_1, \ldots, z_N)$

Notice that, since the h_{kl} above are precisely the Hamiltonian densities for the integrable system associated to the quantum cohomology of M, in the case of a stable Hamiltonian structure of fibration type, the commuting Symplectic Field Theory Hamiltonians actually come (up to a phase) from a Frobenius manifold.

Notice moreover that, thanks to the S^1-symmetry of the Hamiltonian structure, one can deduce a priori that $\mathbf{H}^0 = 0$, so that $H_*(\mathfrak{W}, D^0) = \mathfrak{W}$. In particular, in the case of a trivial S^1-bundle, the Symplectic Field Theory Hamiltonians $\mathbf{h}^1_{kl}$ coincide with all the Hamiltonians of the system associated to the Frobenius manifold $QH^*(M)$, so that we get an integrable system of Hamiltonian PDEs, precisely in the sense of Dubrovin ([13]).

One could rephrase this by saying that genus 0 Symplectic Field Theory of a stable Hamiltonian structure V of fibration type over a closed symplectic manifold M endows the cohomology of the space of periodic Reeb orbits M with a structure of a Frobenius manifold, by defining an integrable system of Hamiltonian PDEs on the loop space of $H^*(M)$.

1.4 General framed symplectic cobordisms

Similarly, in the case of a general framed symplectic cobordism $W = \overrightarrow{V^+V^-}$, after attaching along the boundaries cylindrical ends in the form $V^+ \times \mathbb{R}^+$ and $V^- \times \mathbb{R}^-$, together with the algebras $\mathfrak{W}^+, \mathfrak{W}^-$ (whose formal variables will have a $\pm$ superscript), we consider the space $\mathfrak{D}$ of power series in the variables $p^+_{\gamma^+}$, t_{ij} (associated with differential forms $(\theta_1, \dots, \theta_N)$ such that $\theta_i|_{V^\pm \times \mathbb{R}^\pm} = \theta_i^\pm$) and $\hbar$, with coefficients which are polynomials in $q^-_{\gamma^-}$. Elements in $\mathfrak{W}^\pm$ then act as differential operators on the right/left of $\mathfrak{D}$ after quantizing

$$q^+_{\gamma^+} \mapsto \mu_{\gamma^+} \hbar \overleftarrow{\frac{\partial}{\partial p^+_{\gamma^+}}} \qquad p^-_{\gamma^-} \mapsto \mu_{\gamma_-} \hbar \overrightarrow{\frac{\partial}{\partial q^-_{\gamma^-}}} \tag{1.2}$$

Counting holomorphic curves with cylindrical ends now defines a distinguished element $\mathbf{F} \in \frac{1}{\hbar}\mathfrak{D}$, by requiring that the coefficient of the monomial $\hbar^{g-1} t^I p^{\Gamma^+} q^{\Gamma^-}$, with $t^I = t_{i_1 j_1} \dots t_{i_r j_r}$, $p^{\Gamma^+} = p^+_{\gamma^+_1} \dots p^+_{\gamma^+_{s^+}}$ and $q^{\Gamma^-} = q^-_{\gamma_1} \dots q^-_{\gamma^-_{s^-}}$ is given by the integral

$$\int_{\mathcal{M}_{g,r,\Gamma^-,\Gamma^+}} \mathrm{ev}_1^* \theta_{i_1} \wedge \psi_1^{j_1} \wedge \dots \wedge \mathrm{ev}_r^* \theta_{i_r} \wedge \psi_r^{j_r}$$

where $\psi_i = c_1(L_i)$ is the i-th psi-class (like in Gromov-Witten theory) and this time the integral is over the moduli space of holomorphic curves in W with r marked points and $s^\pm$ positive/negative punctures asymptotically cylindrical over Reeb orbits in $V^\pm$ and realizing, together with the chosen capping surfaces, the homology cycle A in W.

Similarly to what happens with equation (1.1), the geometry of the boundary stratum of one-dimensional moduli spaces translates into a master equation for the potential $\mathbf{F}$ and the Hamiltonians $\mathbf{H}^+$ and $\mathbf{H}^-$ associated to V^+ and V^- respectively,

$$\mathrm{e}^{\mathbf{F}} \overleftarrow{\mathbf{H}^+} - \overrightarrow{\mathbf{H}^-} \mathrm{e}^{\mathbf{F}} = 0 \tag{1.3}$$

where $\overleftarrow{\mathbf{H}^+}$ and $\overrightarrow{\mathbf{H}^-}$ are the operators obtained from $\mathbf{H}^+$ and $\mathbf{H}^-$ by quantization (1.2).

In computing these potentials, it is essential to know the (virtual) dimension of the moduli space $\mathcal{M}^A_{g,r,s^-,s^+}$, that is given in [16] by the index formula

$$\dim \mathcal{M}^A_{g,r,s^-,s^+} = \sum_1^{s^+} \mathrm{CZ}(\gamma_i^+) + \sum_1^{s^-} \mathrm{CZ}(\gamma_k^-) + (n-3)(2-2g-s^+-s^-) + 2c_1^{\mathrm{rel}}(A_{\mathrm{rel}}) + 2r$$

where this time $c_1^{\mathrm{rel}}(A_{\mathrm{rel}})$ is the first relative Chern class of $\mathrm{T}W|_{A_{\mathrm{rel}}}$ and A_{rel} is the relative homology cycle realized by the holomorphic curves in W.

1.5 Evolutionary differential equations for SFT potentials

The first result from [16] we need to recall is the following. Let us assume that W has only a positive end $V \times [0, +\infty)$, and choose what is called a *basic system* $\Delta_1, \dots, \Delta_k$, $\Theta_1, \dots, \Theta_m$ of closed differential forms on W, with cylindrical ends, such that

a) $\Delta_1, \dots, \Delta_k \in H^\bullet(W)$, and the restrictions $\delta_i = \Delta_i|_V$, $i = 1, \dots, l$ for $l \le k$ are independent elements in $\mathrm{Im}(H^\bullet(W) \to H^\bullet(V))$;

b) $\Theta_1, \dots, \Theta_m$ are compactly supported and independent elements in $\mathrm{Ker}(H^\bullet_{\mathrm{comp}}(W) \to H^\bullet(W))$,

c) there exist forms $\theta_1, \dots, \theta_m$ on V and a compactly supported 1-form ρ on $(0, +\infty)$, such that $\Theta_j = \rho \wedge \theta_j$, $j = 1, \dots, m$.

Theorem 1.5.1 ([16]). *Let* $\mathbf{H}$ *be the Hamiltonian associated with the stable Hamiltonian structure* V*. Set*

$$\mathbf{H}_{kl}(t, q, p) = \Big(\frac{\partial \mathbf{H}}{\partial s_{kl}}(\sum_{i=1}^{l} t_{ij}\delta_i + s_{kl}\theta_k, q, p)\Big)\Big|_{s_{kl}=0}$$

$$\mathbf{F}^0(t, p) = \mathbf{F}(\sum t_{ij}\Delta_i, p).$$

Then the potential associated to W *is given by:*

$$\mathrm{e}^{\mathbf{F}(\sum t_{ij}\Delta_i + \sum \tau_{kl}\Theta_k, p)} = \mathrm{e}^{\mathbf{F}^0(t,p)} \prod_{kl} \mathrm{e}^{\tau_{kl}\overleftarrow{\mathbf{H}_{kl}}(t,q,p)},$$

where $\overleftarrow{\mathbf{H}_{kl}}$ *is the operator obtained from* $\mathbf{H}_{kl}$ *by quantization (1.2).*

This result tells us that dependence of $\mathbf{F}$ on the variables associated with compactly supported classes is encoded in the evolution of $\mathbf{F}^0$ along the quantum Hamiltonian system associated with V, i.e. it is controlled by Schrödinger equation. In the classical limit (i.e. when setting $\hbar$ to zero, thus considering the only genus 0 curves), this result can be rephrased in terms of the symplectic geometry of the infinite dimensional phase space of Fourier series we mentioned above, whose space of functions is given by $H_*(\mathfrak{W}^0, D^0)$. The genus 0 potential $\mathbf{f}^0(t, p)$ is then interpreted as the initial datum for an infinite system of compatible Hamilton-Jacobi equations, namely

$$\frac{\partial \mathbf{f}(t, \tau, q, p)}{\partial \tau_{kl}} = \mathbf{h}_{kl}\left(t, \frac{\partial \mathbf{f}(t, \tau, q, p)}{\partial p}, p\right) \qquad k = 1, \dots, m;\ l = 1, 2, \dots$$

In case the commuting Hamiltonians $\mathbf{h}_{kl}$ form an integrable system, one can even use the methods proper of that context for solving the related Hamilton's equation of motion (in the dispersionless case, for instance, the hodograph method, see [13]) and use this solution to attack the above Hamilton-Jacobi problem. In fact, knowing the solution to the equations of motion one can construct the solution to the Hamilton-Jacobi problem by evolving the initial datum $\mathbf{f}^0$ along the motion flow lines (characteristics).

In what follows we also make use of the following theorem, concerning the composition $W = \overrightarrow{V^- V^+}$ of two symplectic cobordisms $W_- = \overrightarrow{V^- V}$ and $W_+ = \overrightarrow{V V^+}$.

Theorem 1.5.2 ([16]). *Let us denote by* $\mathbf{F}_W$*,* $\mathbf{F}_{W_-}$ *and* $\mathbf{F}_{W_+}$ *the SFT-potentials of* W*,* W_- *and* W_+ *respectively. Notice that any cohomology class in* $H^\bullet(W)$ *can be represented by a* *form t which splits into the sum of forms $t_\pm$ with cylindrical ends on $W_\pm$ so that $t_\pm|_V = t_V$. Then:*

$$\mathbf{F}_W(q^-, p^+, t) = \mathbf{F}_{W_-}(q^-, p, t_-) \Diamond \mathbf{F}_{W_+}(q, p^+, t_+)$$

where $\mathrm{e}^{F\Diamond G} = \left(\mathrm{e}^{\overrightarrow{F}}\mathrm{e}^{G}\right)\Big|_{q=0}$ *and $\overrightarrow{F}$ is the operator obtained by quantizing* $p_\gamma = \hbar \mu_\gamma z^{A_\gamma} \overrightarrow{\dfrac{\partial}{\partial q_\gamma}}$ *and $A_\gamma \in H_2(W)$ is the cycle in W formed by the capping surfaces (see again [16] for details) of γ in W_- and W_+.*

Chapter 2

Gromov-Witten invariants of target curves via Symplectic Field Theory

In this chapter we use Symplectic Field Theory techniques to compute at all genera the Gromov-Witten potential of target curves of any genus. This material is mainly taken from the author's paper [32].

Our program is similar to the one via relative Gromov-Witten invariants of Okounkov and Pandharipande ([28]), i.e. we first consider the two building blocks of a Riemann surface (the cap and the pair of pants) and then attach them to obtain the Gromov-Witten potential of the curve Σ_g of genus g. Indeed Symplectic Field Theory can be interpreted as a theory of relative invariants, so the analogy in this sense is strong. Nonetheless our method of computation is not based on Virasoro constraints (as for [28]), but on relations proved in [16] for a very general setting, which makes the computation not only elegant, but even fairly explicit. Moreover, thanks to the surprising emergence of integrable systems from the formalism of Symplectic Field Theory, we can give a beautiful interpretation of many aspects of the full descendants GW-potential in terms of a quantization of the dispersionless KdV hierarchy (after an idea of Eliashberg, [15]).

2.1 Pair of pants potential

In this section we compute the SFT-potential $\mathbf{F}_{\text{pants}}$ for the *pair of pants*, i.e. the completed symplectic cobordism W between $V^- = S^1$ and $V^+ = S^1 \coprod S^1$. W can also be seen as the complex projective line $\mathbb{P}^1$ minus $\{0, 1, \infty\}$, with its standard Kähler structure. This way V^+ corresponds to the circles around, say, 0 and ∞, and V^- to the circle around 1. Call then ϕ the longitude on the Riemann sphere and φ its restriction to V^+; call α the angle on the circle V^- around 1.

Choose the basic system of forms as

$$\Delta_0 = 1 \in H^0(W) \quad \text{restricting to} \quad \delta_0^+ = (1,1) \in H^0(V^+), \quad \delta_0^- = 1 \in H^0(V^-),$$

$$\Delta_1 = \mathrm{d}\phi \in H^1(W) \quad \text{restricting to} \quad \delta_1^+ = (\mathrm{d}\varphi, \mathrm{d}\varphi) \in H^1(V^+), \quad \delta_1^- = 0 \in H^1(V^-)$$

and

$$\Theta_1 \in H^1_{\text{comp}}(W) \quad \text{projecting to} \quad \theta_1^+ = (1,-1) \in H^0(V^+), \quad \theta_1^- = 0 \in H^0(V^-)$$

$$\Theta_2 \in H^2_{\mathrm{comp}}(W) \quad \text{projecting to} \quad \theta_2^+ = (0,0) \in H^1(V^+), \quad \theta_2^- = \mathrm{d}\alpha \in H^1(V^-).$$

The SFT-potential of $V^+ = S^1 \coprod S^1$ (or its symplectization) at all genera is then easily computed from the one of $S^1 \times \mathbb{R}$ (see [16]). What we will need is:

$$\mathbf{H}^+(\sum t_i\delta_i^+ + \sum s_j\theta_j^+) = \frac{1}{\hbar}\left[t_0^2 t_1 + t_0 t_1 s_1 + s_1^2 t_1 + t_1\left(\sum q_k^1 p_k^1 + q_k^2 p_k^2\right) - \frac{\hbar t_1}{12}\right]$$

and

$$\mathbf{H}^-(\sum t_i\delta_i^- + \sum s_j\theta_j^-) = \frac{1}{\hbar}\left[\frac{t_0^2 s_2}{2} + s_2\sum q_k p_k - \frac{\hbar s_2}{24}\right] \tag{2.1}$$

where $\deg t_1 = \deg s_2 = -1$, $\deg t_0 = \deg s_1 = -2$, $\deg p_k^i = \deg p_k = \deg q_k^i = \deg q_k = -2$ and $\deg\hbar = -4$.

Define

$$\mathbf{H}_1^+ := \left.\frac{\partial \mathbf{H}^+}{\partial s_1}\right|_{s_i=0} = \frac{1}{\hbar}t_0 t_1 \tag{2.2}$$

$$\mathbf{H}_2^- := \left.\frac{\partial \mathbf{H}^-}{\partial s_2}\right|_{s_i=0} = \frac{1}{\hbar}\left[\frac{t_0^2}{2} + \sum q_k p_k - \frac{\hbar}{24}\right] \tag{2.3}$$

We will refer to these functions as Hamiltonians. Notice, by the way, that putting

$$u(x) = t_0 + \sum p_k \mathrm{e}^{\mathrm{i}kx} + q_k \mathrm{e}^{-\mathrm{i}kx} \tag{2.4}$$

we have

$$\mathbf{H}_2^- = \frac{1}{\hbar} : \int_{S^1}\left(\frac{u(x)^2}{2} - \frac{\hbar}{24}\right)\mathrm{d}x :$$

where the normal ordering $:\cdot:$ means that the q variables are to be put on the left of the p variables as if, inside the colon symbols, q's and p's all commuted.

We use these Hamiltonians and Theorem 1.5.1 on the initial datum $\mathbf{F}_{\mathrm{pants}}(\sum t_i\Delta_i)$ which is computed directly, by dimension counting. In fact from a combination of the reconstruction theorem for ramified coverings of $\mathbb{P}^1$, the Riemann-Hurwitz theorem and the index formula for the dimension of $\mathcal{M}^A_{g,r,s^+,s^-}$, we get the known relation (see e.g. [27]) between relative Gromov-Witten (or SFT-) potential and Hurwitz numbers

$$\mathbf{F}_{\mathrm{pants}}(\sum t_i\Delta_i) = \sum_g\sum_d\sum_{|\mu^0|,|\mu^1|,|\mu^\infty|=d} H^{\mathbb{P}^1}_{g,d}(\mu^0,\mu^1,\mu^\infty)(p^1)^{\mu^0}(p^2)^{\mu^\infty}q^{\mu^1}\hbar^{g-1} \tag{2.5}$$

where $H^{\mathbb{P}^1}_{g,d}(\mu^0,\mu^1,\mu^\infty)$ is the Hurwitz number counting coverings of $\mathbb{P}^1$ of degree d and genus g, branched only over 0, 1, and ∞ with ramification profile given by μ^0, μ^1 and μ^∞ respectively.

Then, by Theorem 1.5.1 and recalling that $\mathrm{e}^{sx\partial_x}f(x) = f(x\mathrm{e}^s)$, one gets

$$\begin{aligned}\mathbf{F}_{\mathrm{pants}}(\sum t_i\Delta_i + \sum s_j\Theta_j) &= \log\left(\mathrm{e}^{s_2\overrightarrow{\mathbf{H}_2^-}}\,\mathrm{e}^{\mathbf{F}_{\mathrm{pants}}(t)}\,\mathrm{e}^{s_1\mathbf{H}_1^+}\right) = \frac{1}{\hbar}\left[\frac{t_0^2 s_2}{2} + t_0 t_1 s_1 - \frac{\hbar s_2}{24}\right]\\ &+ \sum_g\sum_d\sum_{|\mu^0|,|\mu^1|,|\mu^\infty|=d} H^{\mathbb{P}^1}_{g,d}(\mu^0,\mu^1,\mu^\infty)(p^1)^{\mu^0}(p^2)^{\mu^\infty}(q\mathrm{e}^{s_2})^{\mu^1}\hbar^{g-1}\end{aligned} \tag{2.6}$$

where $\overrightarrow{\mathbf{H}_2^-}$ means the operator obtained from $\mathbf{H}_2^-$ by quantizing $p_k = k\hbar\overrightarrow{\frac{\partial}{\partial q_k}}$.

From this potential, together with the one for the cap (see e.g. [16])

$$\mathbf{F}_{\mathrm{cap}}(t_0 1 + s_2(\rho\wedge \mathrm{d}\phi)) = \frac{1}{\hbar}\left[\frac{t_0^2 s_2}{2} - \frac{s_2}{24} + \frac{1}{\hbar}\mathrm{e}^{s_2}p_1\right] \tag{2.7}$$

and using Theorem 1.5.2 one gets the Gromov-Witten potential for the general genus g Riemann surface. In the next section we illustrate this methods of computation for the simplest case of the elliptic curve E, for which the result is very explicit.

2.2 Gromov Witten potential of E

Let E be a smooth elliptic curve. Consider its pair-of-pants decomposition; it consists of two caps and two pairs of pants. Let U be one of the halves of the torus E when cut along two representatives of the same element in $\pi_1(E)$. U is the symplectic manifold with a contact boundary resulting from attaching a cap to a pair of pants. Its potential is then given by (2.6) and (2.7) by applying Theorem 1.5.2:

$$\begin{aligned}\mathbf{F}_U(\sum t_i\Delta_i + \sum s_j\Theta_j) &= \log\left(\mathrm{e}^{\overrightarrow{\mathbf{F}_{\mathrm{cap}}}(0)}\quad \mathrm{e}^{\mathbf{F}_{\mathrm{pants}}(\sum t_i\Delta_i+\sum s_j\Theta_j)}\right)\Big|_{q=0}\\ &= \log\left(\mathrm{e}^{\overrightarrow{\partial_{q_1}}}\quad \mathrm{e}^{\mathbf{F}_{\mathrm{pants}}(t,s)}\right)\Big|_{q=0}\end{aligned}$$

This results in a substantial simplification of the term containing Hurwitz numbers in (2.6) since putting $q_i = 0$ after derivation with respect to q_1 selects just the terms counting for coverings branched only over 0 and $+\infty$, hence with the same branching number (as it is natural for the potential of the space U). So we get

$$\mathbf{F}_U(\sum t_i\Delta_i + \sum s_j\Theta_j) = \frac{1}{\hbar}\left[\frac{t_0^2 s_2}{2} + t_0t_1s_1 - \frac{\hbar s_2}{24} + \sum\frac{1}{k}p_k^1p_k^2\mathrm{e}^{ks}\right] \tag{2.8}$$

The last step towards the computation of the Gromov-Witten potential of E is performed by gluing two copies of U, one with positive ends and one with negative ends. Let $\tilde{\Delta}_i, \tilde{\Theta}_j \in H^\bullet(E)$ be the obvious extensions of Δ_i, Θ_j from U to the entire E, then the gluing Theorem 1.5.2 takes the form:

$$\mathcal{F}_E\left(\sum t_i\tilde{\Delta}_i + \sum s_j\tilde{\Theta}_j\right) = \left(\mathrm{e}^{\overrightarrow{\mathbf{F}_{U^+}}(t,s)}\quad \mathrm{e}^{\mathbf{F}_{U^-}(0)}\right)\Big|_{q=0}$$

where $\overrightarrow{\mathbf{F}_{U^+}}(t,s)$ is the operator obtained by quantizing $p_k^1 = \hbar k z^k\overrightarrow{\frac{\partial}{\partial q_k^1}}$ and $p_k^2 = \hbar k\overrightarrow{\frac{\partial}{\partial q_k^2}}$.
Explicitly:

$$\begin{aligned}\mathcal{F}_E\left(\sum t_i\tilde{\Delta}_i + \sum s_j\tilde{\Theta}_j\right) =& \frac{1}{\hbar}\left[\frac{t_0^2 s_2}{2} + t_0t_1s_1 - \frac{\hbar s_2}{24}\right] +\\ &+\log\left[\mathrm{e}^{\frac{1}{\hbar}\sum\frac{1}{k}\mathrm{e}^{ks_2}\hbar^2k^2z^k\overrightarrow{\frac{\partial}{\partial q_k^1}}\overrightarrow{\frac{\partial}{\partial q_k^2}}}\ \mathrm{e}^{\frac{1}{\hbar}\sum\frac{1}{k}q_k^1q_k^2}\right]_{q=0}\end{aligned}$$

The differential operator in the second term of the right-hand side can be dealt with by remembering that $\mathrm{e}^{s\partial_x\partial_y}\mathrm{e}^{xy} = \dfrac{\mathrm{e}^{\frac{xy}{1-s}}}{1-s}$, whence (up to an additive constant which is irrelevant for the Gromov-Witten potential)

$$\mathcal{F}_E\left(\sum t_i\tilde{\Delta}_i + \sum s_j\tilde{\Theta}_j\right) = \frac{1}{\hbar}\left[\frac{t_0^2 s_2}{2} + t_0t_1s_1 - \hbar\log(\eta(z\mathrm{e}^{s_2}))\right]$$

where $\eta(q) = q^{1/24}\prod_k(1-q^k)$ is the Dedekind eta function. This agrees with the results of [3].

2.3 Descendent potential of target curves and quantum integrable systems

In the (quite restricted) family of explicitly computed examples (basically the symplectizations of the low-dimensional contact spheres S^{2n+1}), Symplectic Field Theory Hamiltonians show the remarkable property of being integrable, roughly meaning they are part of an infinite number of independent elements in the graded Weyl algebra of p's and q's, forming a graded commutative subalgebra. Indeed, the two simplest cases, S^1 and S^3, correspond to the (quantum) dispersionless versions of the celebrated KdV and Toda hierarchies of integrable PDEs (see e.g. [13]).

Actually this is a general fact ([15]), true for any contact manifold V, and the whole commutative algebra of first integrals of the given $\mathbf{H}_V$ carries the topological information concerning the so called descendants. More precisely we can extend the definition of the functions $\mathbf{H}$ and $\mathbf{F}$, associated as above to symplectic cobordisms, as generating functions for correlators of the type

$$\langle\tau_{p_1}(\Theta_{i_1}),\ldots,\tau_{p_r}(\Theta_{i_r});\gamma_1^-,\ldots,\gamma_{s^-}^-;\gamma_1^+,\ldots,\gamma_{s^+}^+\rangle_g^d =$$
$$\int_{\mathcal{M}^A_{g,r,s^-,s^+}} \mathrm{ev}^*(\Theta_{i_1}\otimes\ldots\otimes\Theta_{i_r}\otimes\gamma_1^-\otimes\ldots\otimes\gamma_{s^-}^-\otimes\gamma_1^+\otimes\ldots\otimes\gamma_{s^+}^+)\wedge c_1(\psi_1)^{p_1}\wedge\ldots\wedge c_1(\psi_r)^{p_r}$$

where now $c_1(\psi_i)$ is the first Chern class of the tautological line bundle over $\mathcal{M}^A_{g,r,s^-,s^+}$ with fibre at $C\in\mathcal{M}^A_{g,r,s^-,s^+}$ equal to the fibre of T^*C at the i-th marked point. We will call the generating functions of these extended correlators $\mathfrak{H}$ and $\mathfrak{F}$ (the first for the cylindrical case, the second for the general cobordism) and they will depend on the variables (p,q,t_{i_j,p_k}) with $t_{i_j,0} = t_{i_j}$ as before.

Theorems 1.5.1 and 1.5.2 extend then to the case of descendants, i.e. they hold true for the extended functions $\mathfrak{H}$ and $\mathfrak{F}$. Even more fundamentally, the structure equation $[\mathbf{H},\mathbf{H}]=0$ extends to $[\mathfrak{H},\mathfrak{H}]=0$ ($\mathbf{H}$ and $\mathfrak{H}$ are here evaluated on closed forms, so the terms $\mathrm{d}\mathbf{H}$ and $\mathrm{d}\mathfrak{H}$ drop), ensuring that the Hamiltonians appearing in Theorem 1.5.1 actually commute, forming a quantum integrable system.

As an example we determine the integrable system involved in the computation of the Gromov-Witten potential of target curves with descendants of $\omega\in H^2(\Sigma_g)$ (in what follows we put even $t_0=0$ for simplicity). We already know that the relevant cylindrical cobordism is just the cylinder $V = S^1 \times \mathbb{R}$. Call, as above, t_0 and s_2 the components along 1 and $\mathrm{d}\phi$ of a form in $H^\bullet(V)$, then its potential without descendants $\mathbf{H}_0$ is given by (2.1). With a little dimension counting over the moduli space $\mathcal{M}^A_{g,r,s^-,s^+}$ one can also compute the first descendant potential with respect to $\mathrm{d}\phi$ (i.e. the part of $\mathfrak{H}$ depending just on t_0 and linearly on $s_{2,1}$) that, with the notation of (2.4), takes the form

$$s_{2,1}\,\mathbf{H}_1 = s_{2,1}\,\frac{1}{\hbar} : \int_{S^1} \left(\frac{u(x)^3}{6} - \hbar \frac{u(x)}{24} \right) \mathrm{d}x :$$

This (quantum) Hamiltonian, corresponding to a quantum dispersionless KdV system, is sufficient to determine uniquely the whole integrable system (notice that it was not so for $\mathbf{H}_0$, since any local Hamiltonian commutes with it), i.e. the algebra of commuting Hamiltonians of Theorem 1.5.1. The explicit determination of this commutative algebra has recently been performed in an elegant context of fermionic calculus by Pogrebkov in [31]. Nonetheless the topological information about the target cobordism is encoded also in an explicit choice of a basis of such (super-)commutative algebra.

Such a basis is given ([15]) by

$$\mathbf{H}_{n-2} = \frac{1}{\hbar}\frac{1}{n!} \left.\frac{\partial^n \mathcal{H}(z)}{\partial z^n}\right|_{z=0} \tag{2.9}$$

in terms of the generating function

$$\mathcal{H}(z) = \frac{1}{\mathcal{S}(\sqrt{\hbar} z)} : \int_{S^1} \mathrm{e}^{z\mathcal{S}(\mathrm{i}\sqrt{\hbar} z \partial_x)\, u(x)} \mathrm{d}x : \qquad \text{with} \qquad \mathcal{S}(t) = \frac{\sinh \frac{t}{2}}{\frac{t}{2}} \tag{2.10}$$

Here $: \cdot :$ means that q and p variables are to be normal ordered (q's are to be put on the left) as if, inside the colon symbols, q's and p's all commuted. As we show in the next section, this generating function can be obtained from the work of Okounkov and Pandharipande, directly by expressing the operator

$$\mathcal{E}_0(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{\mathrm{e}^{\sqrt{\hbar} z k}}{\hbar} : \psi_k \psi_k^* : + \frac{1}{\mathrm{e}^{\sqrt{\hbar} z/2} - \mathrm{e}^{-\sqrt{\hbar} z/2}} \tag{2.11}$$

of [27], acting on the fermionic Fock space (see below) and relevant for the generating function of 1-point invariants of $\mathbb{P}^1$ relative to 0 and ∞, in terms of vertex operators via the "boson-fermion" correspondence (see e.g. [26],[22]).

The function $\mathcal{S}(t)$, appearing repeatedly also in the work of Okounkov and Pandharipande, emerges in [31] directly from an integrable systems context (although there a different basis of Hamiltonians for the commutative algebra of symmetries is chosen) and is involved in the topological picture via the Schrödinger equation of Theorem 1.5.1. For instance, let's write this equation for the simplest case of the descendant SFT-potential for the cap:

$$\mathfrak{F}_{\mathrm{cap}}(s_{2,k}) = \log \left(\mathrm{e}^{\mathfrak{F}_{\mathrm{cap}}(0)} \prod_{n=0}^{\infty} \mathrm{e}^{s_{2,n} \overleftarrow{\mathbf{H}_n}} \right) \tag{2.12}$$

In the next sections, after recalling its fundamental ideas, we make use of fermionic calculus to obtain the fermionic expression of the generating function (2.10) and compute explicitly $\mathfrak{F}_{\mathrm{cap}}(s_{2,k})$.

2.4 Fermionic calculus

Following [26], notice that the Weyl algebra of differential operators of the form

$$\sum c_{\alpha_1\alpha_2\ldots\beta_1\beta_2\ldots} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \left(\frac{\partial}{\partial p_1}\right)\left(\frac{\partial}{\partial p_2}\right)\ldots,$$

involved in the Symplectic Field Theory of target curves, is a representation of the algebra $\mathcal{B}$ generated by the *bosons* $na_n := q_n$ and $a_n^* := p_n$ (with the usual commutation relations) on the *bosonic Fock space* $\mathbb{C}[[p]] := \mathbb{C}[[p_1, p_2, \ldots]] = \mathcal{B} \cdot 1$. Notice that the element $1 \in \mathbb{C}[[p]]$ (called the *bosonic vacuum state*) is annihilated by a_n, for every n. In particular $\mathbb{C}[[p]]$ has the basis

$$\{a_{m_1} \ldots a_{m_r} \cdot 1 \mid 0 < m_1 < \ldots < m_r\}.$$

Next we introduce another algebra, the Clifford algebra $\mathcal{A}$ generated by the *fermions* $\{\psi_n, \psi_n^*\}_{m,n\in\mathbb{Z}+1/2}$ with the anti-commutation relations

$$\{\psi_m, \psi_n\} = 0, \qquad \{\psi_m^*, \psi_n^*\} = 0, \qquad \{\psi_m, \psi_n^*\} = \hbar\delta_{m,n}.$$

As for the bosonic case, we represent $\mathcal{A}$ on the *fermionic Fock space* $\mathcal{F} := \mathcal{A} \cdot |\mathrm{vac}\rangle$, where the *fermionic vacuum state* $|\mathrm{vac}\rangle$ is annihilated by ψ_k with $k < 0$ and ψ_l^* with $l > 0$. In what follows we will also write $\langle \mathrm{vac}| \cdot \mathcal{A}$ to denote the dual Fock space $\mathcal{F}^*$ (and similarly for its elements).

Besides the bosonic normal ordering of p and q operators, which we already defined, we introduce the following fermionic normal ordering:

$$: \psi_m \psi_n^* := \begin{cases} \psi_m \psi_n^* & \text{if } m > 0 \text{ or if } n > 0 \\ -\psi_n^* \psi_m & \text{if } m < 0 \text{ or if } n < 0 \end{cases}$$

The space $\mathcal{F}$ (and its dual) is graded according to the so called *charge* l as $\mathcal{F} = \bigoplus_{l\in\mathbb{Z}} \mathcal{F}_l$ where $\mathcal{F}_l$ is the span of

$$\{\psi_{m_r} \ldots \psi_{m_1} \psi_{n_1}^* \ldots \psi_{n_s}^* |\mathrm{vac}\rangle \mid m_1 < \ldots < m_r,\, n_1 < \ldots < n_s < 0,\, r - s = l\}.$$

The following is an important and well known result (proven e.g. in [26]) usually called the boson-fermion correspondence.

Theorem 2.4.1. *There is an isomorphism* $\Phi : \mathcal{F}_0 \to \mathbb{C}[[p]]$ *of the form*

$$\Phi(|u\rangle) := \langle \mathrm{vac}| \mathrm{e}^{K(p)} |u\rangle.$$

where $K(p) := \frac{1}{\hbar}\sum_{n=1}^{\infty} \frac{p_n}{n} \sum_{j\in\mathbb{Z}+1/2} : \psi_j \psi_{j+n}^* :$
At the level of operators the isomorphism is described by

$$\Phi^{-1} q_n \Phi = \sum_{j\in\mathbb{Z}+1/2} : \psi_j \psi_{j+n}^* :$$

$$\Phi^{-1} p_n \Phi = \sum_{j\in\mathbb{Z}+1/2} : \psi_j \psi_{j-n}^* :$$

Example 2.4.2. Using the formula $\mathrm{e}^A B \mathrm{e}^{-A} = \mathrm{e}^{\mathrm{ad}_A} B$ for the adjoint action of a Lie algebra on itself we get

$$\mathrm{e}^{K(p)}\psi_n \mathrm{e}^{-K(p)} = \psi_n + \frac{p_1}{\hbar}\psi_{n-1} + \left(\frac{p_2}{\hbar} + \frac{p_1^2}{2\hbar^2}\right)\psi_{n-2} + \dots$$

$$\mathrm{e}^{K(p)}\psi_n^* \mathrm{e}^{-K(p)} = \psi_n^* - \frac{p_1}{\hbar}\psi_{n+1}^* + \left(-\frac{p_2}{\hbar} + \frac{p_1^2}{2\hbar^2}\right)\psi_{n+2}^* + \dots$$

and, using these, we can compute for instance

$$\begin{aligned}\Phi(\psi_{3/2}\psi_{-1/2}^*|\mathrm{vac}\rangle) =&\langle\mathrm{vac}|\mathrm{e}^{K(p)}\psi_{3/2}\psi_{-1/2}^*|\mathrm{vac}\rangle \\ =&\langle\mathrm{vac}|\left(\psi_{3/2} + \frac{p_1}{\hbar}\psi_{1/2} + \left(\frac{p_2}{\hbar} + \frac{p_1^2}{2\hbar^2}\right)\psi_{-1/2} + \dots\right)\left(\psi_{-1/2}^* + \dots\right)|\mathrm{vac}\rangle \\ =&\frac{p_2}{2} + \frac{p_1^2}{2\hbar}\end{aligned}$$

and similarly

$$\Phi(\psi_{1/2}\psi_{-3/2}^*|\mathrm{vac}\rangle) = \frac{p_2}{2} - \frac{p_1^2}{2\hbar}$$

△

Now consider the following *vertex operator* acting (on the right) on $\mathbb{C}[[p]]$

$$X(z_1, z_2) := \mathrm{e}^{\sum_{j=1}^\infty \sqrt{\hbar}(z_1^j - z_2^j)\overleftarrow{\frac{\partial}{\partial p_j}}}\,\mathrm{e}^{-\sum_{j=1}^\infty \frac{1}{\sqrt{\hbar}j}(z_1^{-j} - z_2^{-j})p_j}$$

with $z_1, z_2 \in \mathbb{C}$. Then the above theorem gives as a corollary

$$\frac{1}{\hbar} : \psi(z_1)\psi^*(z_2) : \xrightarrow{\Phi} \frac{1}{z_1 - z_2}\left(X(z_1, z_2) - 1\right) \tag{2.13}$$

where $\psi(z) = \sum_{n\in\mathbb{Z}+1/2}\psi_n z^{n-1/2}$ and $\psi^*(z) = \sum_{n\in\mathbb{Z}+1/2}\psi_n^* z^{-n-1/2}$

Getting back to the generating function (2.10) we make the following important observation:

$$\overleftarrow{\mathcal{H}(z)} = \frac{1}{\mathcal{S}(\sqrt{\hbar}z)}\int_{S^1} X\left(\mathrm{e}^{\mathrm{i}\left(x - \mathrm{i}\frac{\sqrt{\hbar}z}{2}\right)}, \mathrm{e}^{\mathrm{i}\left(x + \mathrm{i}\frac{\sqrt{\hbar}z}{2}\right)}\right)\mathrm{d}x$$

whence, by applying (2.13) and computing the residue, we get the fermionic expression for the generating function of the quantum dispersionless KdV Hamiltonians

Theorem 2.4.3.

$$\overleftarrow{\mathcal{H}(z)} \xrightarrow{\Phi} \sqrt{\hbar}z\mathcal{E}_0(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{\sqrt{\hbar}z\mathrm{e}^{\sqrt{\hbar}zk}}{\hbar} : \psi_k\psi_k^* : + \frac{\sqrt{\hbar}z}{\mathrm{e}^{\sqrt{\hbar}z/2} - \mathrm{e}^{-\sqrt{\hbar}z/2}} \tag{2.14}$$

This way the equivalence of our generating function of 1-point SFT-invariants of the tube and the one for 1-point GW-invariants relative to 0 and ∞ (derived by Okounkov and

Pandharipande in [27] using the Gromov-Witten/Hurwitz correspondence), is proven. From (2.14) we get

$$\overleftarrow{\mathbf{H}}_n \overset{\Phi}{\longrightarrow} \frac{\hbar^{n/2}}{(n+1)!}\left[\sum_{k\in\mathbb{Z}+1/2}\frac{k^{n+1}}{\hbar}:\psi_k\psi_k^*:+(1-2^{-(n+1)})\zeta(-(n+1))\right] \tag{2.15}$$

where ζ is the Riemann zeta function. The peculiar diagonal form of the right-hand side ensures the required commutativity of the Hamiltonians. Now we can use this fermionic formalism to compute the invariants of $\mathbb{P}^1$ (or, in principle, of any compact Riemann surface) via Theorems 1.5.1 and 1.5.2.

2.5 Descendant potential of $\mathbb{P}^1$

First of all we rewrite equation (2.12) for the descendant SFT-potential of the cap in the language of fermionic calculus of the previous section. We get

$$\mathrm{e}^{\mathfrak{F}_{\mathrm{cap}}(s_{2,k})} = \mathrm{e}^{\sum_n s_{2,n}\frac{\hbar^{n/2}}{(n+1)!}\left[\sum_{k\in\mathbb{Z}+1/2}\frac{k^{n+1}}{\hbar}:\psi_k\psi_k^*:+(1-2^{-(n+1)})\zeta(-(n+1))\right]}\ \mathrm{e}^{\sum_{j\in\mathbb{Z}+1/2}:\psi_j\psi_{j-1}^*:}\ |\mathrm{vac}\rangle$$

One is now to compute the right hand side and then use Theorem 1.5.2 to get the descendant potential of $\mathbb{P}^1$. If we restrict to the case of maps with fixed degree the computations can be carried out very explicitly. Remember that both the SFT and Gromov-Witten potentials are power series in z with coefficients the potentials for maps with fixed degree. We write

$$\mathcal{F}_{\mathbb{P}^1}(s_{2,k},z) = \sum_i \mathcal{F}_{\mathbb{P}^1,i}(s_{2,k})z^i$$

$$\mathfrak{F}_{\mathrm{cap}}(s_{2,k},p\mapsto zp) = \sum_i \mathfrak{F}_{\mathrm{cap},i}(s_{2,k},p)z^i$$

Then Theorems 1.5.1 and 1.5.2 give:

DEGREE 0:

$$\mathrm{e}^{\mathcal{F}_{\mathbb{P}^1,0}(s_{2,k})} = \mathrm{e}^{\overrightarrow{\mathfrak{F}}_{\mathrm{cap},0}(s_{2,k},p)}\ \mathrm{e}^{\mathfrak{F}_{\mathrm{cap}}(0,q)}$$

$$\mathrm{e}^{\mathfrak{F}_{\mathrm{cap},0}(s_{2,k},p)} = 1\ \mathrm{e}^{\sum_n s_{2,n}\overleftarrow{\mathbf{H}}_n}$$

and using (2.15) we immediately get (see [19],[29])

$$\mathfrak{F}_{\mathrm{cap},0}(s_{2,k},p) = \sum_n s_{2,n}\frac{\hbar^n}{(n+1)!}(1-2^{-(n+1)})\zeta(-(n+1))$$

$$\mathcal{F}_{\mathbb{P}^1,0}(s_{2,k}) = \sum_n s_{2,n}\frac{\hbar^n}{(n+1)!}(1-2^{-(n+1)})\zeta(-(n+1))$$

DEGREE 1:

$$\mathcal{F}_{\mathbb{P}^1,1}(s_{2,k}) = \overrightarrow{\mathfrak{F}}_{\mathrm{cap},1}(s_{2,k},p)\ \mathrm{e}^{\mathfrak{F}_{\mathrm{cap}}(0,q)}$$

$$\mathfrak{F}_{\mathrm{cap},1}(s_{2,k},p)\ \mathrm{e}^{\mathfrak{F}_{\mathrm{cap},0}(s_{2,k},p)} = \frac{1}{\hbar}p_1\ \mathrm{e}^{\sum_n s_{2,n}\overleftarrow{\mathbf{H}}_n}$$

and passing to the fermions the second equation becomes

$$\Phi^{-1}(\mathfrak{F}_{\mathrm{cap},1}(s_{2,k},p)) = \frac{1}{\hbar}\mathrm{e}^{\sum_k\left(\sum_n s_{2,n}\frac{\hbar^{(n-2)/2}}{(n+1)!}k^{n+1}\right):\psi_k\psi_k^*:}\ \psi_{1/2}\psi_{-1/2}\ |\mathrm{vac}\rangle$$

Since

$$\left[\sum_k\left(\sum_n s_{2,n}\frac{\hbar^{(n-2)/2}}{(n+1)!}k^{n+1}\right) : \psi_k\psi_k^* : ,\ \psi_{1/2}\psi_{-1/2}\right] = \sum_n s_{2,2n}\frac{\hbar^n}{2^{2n}(2n+1)!}\ \psi_{1/2}\psi_{-1/2},$$

we get

Theorem 2.5.1.

$$\mathfrak{F}_{\mathrm{cap},1}(s_{2,k},p) = \frac{1}{\hbar}\mathrm{e}^{\sum_n s_{2,2n}\frac{\hbar^n}{2^{2n}(2n+1)!}}\ p_1$$

$$\mathcal{F}_{\mathbb{P}^1,1}(s_{2,k}) = \frac{1}{\hbar}\mathrm{e}^{\sum_n s_{2,2n}\frac{\hbar^n}{2^{2n}(2n+1)!}}$$

The second equation coincides with the result found by Pandharipande in [29] using Toda equations.

DEGREE 2:

$$\left(\mathcal{F}_{\mathbb{P}^1,1}(s_{2,k})\right)^2 + 2\mathcal{F}_{\mathbb{P}^1,2}(s_{2,k}) = \left[\left(\overrightarrow{\mathfrak{F}}_{\mathrm{cap},1}(s_{2,k},p)\right)^2 + 2\,\overrightarrow{\mathfrak{F}}_{\mathrm{cap},2}(s_{2,k},p)\right]\ \mathrm{e}^{\mathfrak{F}_{\mathrm{cap}}(0,q)}$$

$$\left[(\mathfrak{F}_{\mathrm{cap},1}(s_{2,k},p))^2 + 2\mathfrak{F}_{\mathrm{cap},2}(s_{2,k},p)\right]\ \mathrm{e}^{\mathfrak{F}_{\mathrm{cap},0}(s_{2,k},p)} = \frac{1}{\hbar^2}p_1^2\ \mathrm{e}^{\sum_n s_{2,n}\overleftarrow{\mathbf{H}}_n}$$

and passing to fermions the second equation becomes (see Example 2.4.2)

$$\begin{aligned}&\Phi^{-1}((\mathfrak{F}_{\mathrm{cap},1}(s_{2,k},p))^2 + 2\mathfrak{F}_{\mathrm{cap},2}(s_{2,k},p)) =\\ &\frac{1}{\hbar}\mathrm{e}^{\sum_k\left(\sum_n s_{2,n}\frac{\hbar^{(n-2)/2}}{(n+1)!}k^{n+1}\right):\psi_k\psi_k^*:}\ (\psi_{3/2}\psi^*_{-1/2} - \psi_{1/2}\psi^*_{-3/2})\ |\mathrm{vac}\rangle\end{aligned}$$

Since

$$\begin{aligned}&\left[\sum_k\left(\sum_n s_{2,n}\frac{\hbar^{(n-2)/2}}{(n+1)!}k^{n+1}\right) : \psi_k\psi_k^* : ,\ (\psi_{3/2}\psi^*_{-1/2} - \psi_{1/2}\psi^*_{-3/2})\right] =\\ &\sum_n s_{2,n}\frac{\hbar^n/2}{(n+1)!}\left(\left(\frac{3}{2}\right)^{n+1} - \frac{1}{(-2)^{n+1}}\right)\psi_{3/2}\psi^*_{-1/2}\\ &-\sum_n s_{2,n}\frac{\hbar^n/2}{(n+1)!}\left(\frac{1}{2^{n+1}} - \left(\frac{3}{2}\right)^{n+1}\right)\psi_{1/2}\psi^*_{-3/2},\end{aligned}$$

we get

$$\begin{aligned}&(\mathfrak{F}_{\mathrm{cap},1}(s_{2,k},p))^2 + 2\mathfrak{F}_{\mathrm{cap},2}(s_{2,k},p) =\\ &\frac{1}{\hbar}\mathrm{e}^{\sum_n s_{2,n}\frac{\hbar^n/2}{(n+1)!}\left(\left(\frac{3}{2}\right)^{n+1}-\frac{1}{(-2)^{n+1}}\right)}\left(p_2 + \frac{p_1^2}{2\hbar}\right) - \frac{1}{\hbar}\mathrm{e}^{\sum_n s_{2,n}\frac{\hbar^n/2}{(n+1)!}\left(\frac{1}{2^{n+1}}-\left(\frac{3}{2}\right)^{n+1}\right)}\left(p_2 - \frac{p_1^2}{2\hbar}\right)\end{aligned}$$

and finally

Theorem 2.5.2.

$$\mathcal{F}_{\mathbb{P}^1,2}(s_{2,k}) =$$
$$\frac{1}{2\hbar^2}\left[\frac{1}{2}\mathrm{e}^{\sum_n s_{2,n}\frac{\hbar^n/2}{(n+1)!}\left(\left(\frac{3}{2}\right)^{n+1}-\frac{1}{(-2)^{n+1}}\right)} - \frac{1}{2}\mathrm{e}^{\sum_n s_{2,n}\frac{\hbar^n/2}{(n+1)!}\left(\frac{1}{2^{n+1}}-\left(\frac{3}{2}\right)^{n+1}\right)} + \mathrm{e}^{\sum_n s_{2,2n}\frac{\hbar^n}{2^{2n-1}(2n+1)!}}\right]$$

This process can be carried on to even higher degree, with more and more struggle, but it appears, already at degree 2, to be fairly more efficient than the method of Virasoro constraints of [17], used in [28].

Chapter 3

Gromov-Witten theory of polynomial $\mathbb{P}^1$-orbifolds

In this chapter we give a procedure for computing the genus g orbifold Gromov-Witten potential of an orbicurve C, when only the even (ignoring degree shifting) orbifold cohomology is considered, in terms of certain Hurwitz numbers. This material is mainly taken from the author's paper [33].

Orbifold Gromov-Witten theory, like its smooth version, can be approached either in the differentiable or in the algebraic language. We chose, coherently with with the symplectic environment, to use the first one, introduced by Chen and Ruan ([8],[9]). The main reference for the algebraic geometric approach is by Abramovich, Graber and Vistoli, in [1],[2].

In general the orbicurve Gromov-Witten potential is a power series of the orbifold cohomology variables $t_0, \ldots, t_l, \mathrm{e}^{t_{l+1}}$ (here l is the dimension of the even orbifold cohomology of the orbicurve). Our computation will use a very mild extension of Symplectic Field Theory where we allow orbifold singularities of the target cobordisms. Of course such a extension, in its full generality, would need a thorough study of the moduli space geometry it gives rise, in order to fix the foundations, which is still to be completed even in the ordinary smooth case. However here we consider just the case of 2-dimensional target cobordisms with $\mathbb{Z}_k$ isolated singularities (in fact punctured orbicurves), where the slight modifications to the theory (mainly the index formula for the dimension of the moduli space of maps and the grading of the variables) and its main results can be guessed easily.

Once we have described the procedure to obtain this "partial" potential for C as a series, we restrict to the case of $\mathbb{P}^1_{\alpha_1,\ldots,\alpha_a}$ (the complex projective line with a orbifold points with singularities $\mathbb{Z}_{\alpha_i}$) and we ask when this series truncates. The answer turns out to be that the Frobenius structure is polynomial (in $t_1, \ldots, t_l, \mathrm{e}^{t_{l+1}}$) exactly for any $\mathbb{P}^1_{\alpha_1,\alpha_2}$ (see [25]) and for $\mathbb{P}^1_{2,2,l-2}$ ($l \geq 4$), $\mathbb{P}^1_{2,3,3}$, $\mathbb{P}^1_{2,3,4}$ and $\mathbb{P}^1_{2,3,5}$.

Finally, after extending the result about SFT of Hamiltonian structures of fibration type to the case of an orbifold base M, we compute the SFT-potential of a general Seifert fibration over $\mathbb{P}^1_{a,b,c}$ with different contact or Hamiltonian strucures given in the usual way by the fibration itself. This potential will involve, as expected, the integrable systems associated to the rational Gromov-Witten theory of the base. This process can actually be carried on, along the lines of section 2.9.3 of [16], to extract symplectic and contact invariants of higher dimensional manifolds.

3.1 Symplectic Field Theory of punctured Riemann surfaces

In this section we prove a general formula for the SFT-potential of a two-dimensional symplectic cobordism S obtained from a genus g' Riemann surface $\Sigma_{g'}$ by removing small discs around a points $z_1, \ldots, z_a \in \Sigma_{g'}$ (see also [32]). We will consider it as a framed symplectic cobordims between the disjoint union of m copies of S^1 and the empty set and we will consider only the dependence on the even cohomology classes (i.e those represented by 0-forms and 2-forms). The potential will be expressed in terms of Hurwitz numbers $H^{\Sigma_{g'}}_{g,d}(\mu_1, \ldots, \mu_a)$, i.e. the number coverings of $\Sigma_{g'}$ of genus g and degree d, branched only over $z_1, \ldots, z_a$ with ramification profile $\mu_1, \ldots, \mu_a$ respectively.

Lemma 3.1.1.

$$\begin{aligned}\mathbf{F}_S(t_0, s_2, p^1, \ldots, p^a) =& \frac{1}{\hbar}\frac{t_0^2 s_2}{2} - \frac{s_2}{24} \\ &+ \sum_{g=0}^{\infty}\sum_{d=0}^{\infty}\sum_{|\mu_1|,\ldots,|\mu_a|=d} \mathrm{e}^{ds_2} H^{\Sigma_{g'}}_{g,d}(\mu_1, \ldots, \mu_a)(p^1)^{\mu_1}\ldots(p^a)^{\mu_a}\hbar^{g-1}\end{aligned}$$

where t_0, s_2 are the variables associated to $\Delta_0 = 1 \in H^0(S)$ and $\Theta_2 = \omega \in H^2_{\mathrm{comp}}(S)$ respectively.

Proof. Recall the index theorem for the (virtual) dimension of $\mathcal{M}^A_{g,r,s^+,s^-}$, whence:

$$\dim \mathcal{M}^A_{g,r,s}(S) = 2\left(\sum_{i=1}^{a} s_i + 2g - 2\right) + 2r + 2d(2 - 2g' - a)$$

where s_i is the number of punctures asymptotically mapped to z_i and $s_i = d - \sum_{k=1}^{s_i}(e^i_k - 1)$ where e^i_k is the multiplicity of the k-th puncture mapped to z_i. Let us write the potential for $t_0 = s_2 = 0$: then the relevant component of the moduli space must be zero-dimensional. The above formula, together with Riemann-Hurwitz theorem for branched coverings, ensures then that we have to count curves with no marked points ($r = 0$) and no branch values other then $z_1, \ldots, z_n$, hence

$$\mathbf{F}_S(t_0 = 0, s_2 = 0, p^1, \ldots, p^a) = \sum_{g=0}^{\infty}\sum_{d=0}^{\infty}\sum_{|\mu_1|,\ldots,|\mu_a|=d} H^{\Sigma_{g'}}_{g,d}(\mu_1, \ldots, \mu_a)(p^1)^{\mu_1}\ldots(p^a)^{\mu_a}\hbar^{g-1}$$

It now remains to apply Theorem 2.7.1 of [16], making $\exp(\mathbf{F}_S(t_0 = 0, s_2 = 0, p^1, \ldots, p^a))$ evolve through Schrödinger equation with KdV Hamiltonian $\mathbf{H} = \frac{t_0^2}{2} - \frac{\hbar}{24} + \sum p^i_k q^i_k$ to get the desired formula. ◻

3.2 Gromov-Witten invariants of orbicurves

We now plan to use Lemma 3.1.1 and the gluing theorem for composition of cobordisms (Theorem 2.5.3 of [16]) to obtain the orbifold Gromov-Witten potential of a genus g' Riemann surface with orbifold points. In order to do that we will need to consider the mild generalization of Symplectic Field Theory we referred to in the Introduction. Namely we allow the target 2-dimensional symplectic cobordisms C to have a finite number of isolated codimension-2 $\mathbb{Z}_k$ singularities (see e.g. [8][9] for a review of orbifold geometry). The cobordism C can then be identified with a punctured orbicurve. We will denote by n the number of punctures, by a the number of singular points and by $\alpha_1, \ldots, \alpha_a$ the orders of the singularities of C. The relevant moduli space of holomorphic maps will inherit both the characteristics of Chen and Ruan's space of orbicurves ([9]) and of the usual SFT's space of curves with punctures asymptotic to Reeb orbits at $\pm\infty$ ([16]). We will denote by $\mathcal{M}^d_{g,r,s^+,s^-}(C,J,\mathbf{x})$ the moduli space of holomorphic genus g orbicurves in C of degree d with r marked points, $s^\pm$ positive/negative punctures, which are of type $\mathbf{x}$ in the sense of [9] (here $\mathbf{x}$ is a connected component in the inertia orbifold $\tilde{C}$). In particular, smoothness of the cylindrical ends of the target cobordism ensures that the main operations on $\mathcal{M}^d_{g,r,s^+,s^-}(C,J,\mathbf{x})$, first of all its compactification, can be performed completely analogously to the smooth case (with appearance of n-story stable curves and the usual rich structure of the boundary).

Let us write, for instance and future reference, the index formula for the dimension of this generalized moduli space:

$$\dim \mathcal{M}^d_{g,r,s^+,s^-}(C,J,\mathbf{x}) = 2(2g+s^+ + s^- - 2) + 2d\, c_1^{\mathrm{rel}}(C) + 2r - 2\iota(\mathbf{x})$$

where

$$c_1^{\mathrm{rel}}(C) = (2-2g-n+\sum_1^a \frac{1-\alpha_i}{\alpha_i})$$

is the first Chern class of C, relative to the boundary, and $\iota(\mathbf{x})$ is the degree shift of the connected component $\mathbf{x}$ of $\tilde{C}$ (see [8] for details).

Similarly, when one considers the Weyl and Poisson graded algebras of SFT (see [16] for details) where the Gromov-Witten-like potentials are defined, the grading for the p and q variables, together with the cohomological ones (t$=\sum t_i\theta_i$, where θ_i is an homogeneous element in $\Omega^*(\tilde{C})$), will be given by formulas that are totally analogous to the ones for the smooth case (see [16])

$$\begin{aligned}
\deg p_\gamma &= +CZ(\gamma) - 2\\
\deg q_\gamma &= -CZ(\gamma) - 2\\
\deg t_i &= \deg_{\mathrm{orb}} \theta_i - 2\\
\deg z &= -2c_1(C)
\end{aligned}$$

but take into account the effect of the singularities on the orbifold cohomological degree and on the trivialization of the relevant symplectic bundles, giving rise to rational Conley-Zehnder indices. We will give below explicit formulas for the specific cases we are going to use. With this grading, the property of the SFT-potential of a cobordism of being homogeneous of degree 0, is preserved even in this singular case.

The basic building blocks of our construction for the Gromov-Witten potential of an orbicurve are the potentials for the punctured Riemann surface (Lemma 3.1.1) and the α-orbifold cap $\mathbb{C}/\mathbb{Z}_\alpha$. Let us compute this last missing element, which we see as a singular symplectic cobordism between S^1 and the empty manifold. Consider orbifold cohomology classes of the orbifold cap, which we denote $t_0, t_1, \ldots, t_{\alpha-1}$, where $t_i \in H^{2i/\alpha}_{\mathrm{orb}}(\mathbb{C}/\mathbb{Z}_\alpha)$. The grading for these variables is given, accordingly, by $\deg t_i = \frac{2i}{\alpha} - 2$. Moreover, the natural trivialization of the tangent bundle of the cap along the circle boundary gives a fractional CZ-index of $1/\alpha$ for the simple orbit (one actually gets just a natural trivialization for the α-fold covering of the cap which gives $CZ(\alpha S^1) = 1$), which translates to $\deg p_k = -\frac{2k}{\alpha} - 2$. In order to compute the explicit form of the SFT-potential of $\mathbb{C}/\mathbb{Z}_\alpha$, which we denote $\mathbf{F}_\alpha(t_0, \ldots, t_{\alpha-1}, p, \hbar)$, we proceed the following way. Recall that $\mathbf{F}_\alpha = \sum \hbar^{g-1}\mathbf{F}_{\alpha,g}$ must be homogeneous of degree 0, with $\deg \hbar = -4$. Since all our variables have negative degree we deduce that the only nonzero terms appear for genus $g = 0$. Here we have

$$\mathbf{F}_\alpha = \frac{1}{\hbar} \sum_{\substack{i_0,\ldots,i_{\alpha-1} \\ \sum(\alpha-k)i_k = 2\alpha}} A_{i_0\ldots i_{\alpha-1}} t_0^{i_0} \ldots t_{\alpha-1}^{i_{\alpha-1}} + \frac{1}{\hbar}\sum_{j=1}^{\alpha} \sum_{\substack{i_1,\ldots,i_{\alpha-1} \\ \sum(\alpha-k)i_k = \alpha-j}} B_{i_1\ldots i_{\alpha-1};j}\, t_1^{i_1} \ldots t_{\alpha-1}^{i_{\alpha-1}} \frac{p_j}{j}$$

One way of determining the coefficients A and B is by gluing any two orbifold caps along their boundary S^1 and use Theorems 2.7.1 and 2.5.3 of [16] to compute the Gromov-Witten potential $\mathbf{F}_{\alpha_1,\alpha_2}(t_0, t_{1,1}, \ldots, t_{1,\alpha_1-1}, t_{2,1}, \ldots, t_{2,\alpha_2-1}, s)$ of the resulting $\mathbb{P}^1_{\alpha_1,\alpha_2}$ (here, of course, $s \in H^2(\mathbb{P}^1_{\alpha_1,\alpha_2})$). This potential was already computed by Milanov and Tseng in [25], who showed its relation with Carlet's extended bigraded Toda hierarchy [6] and, hence, with extended affine Weyl groups for the root systems A_l (see [12]). Alternatively, one can impose WDVV equations on these potentials and determine the desired coefficients. (this is, by the way, a very efficient way to compute GW-invariants of $\mathbb{P}^1_{\alpha_1,\alpha_2}$).

Example 3.2.1. The SFT-potential of $\mathbb{C}/\mathbb{Z}_2$ has the form

$$\mathbf{F}_2(t_0, t_1, p) = \frac{1}{\hbar}\left[A t_0 t_1^2 + B t_1^4 + C t_1 p_1 + D p_2\right]$$

and a comparison with the genus 0 GW-potential of $\mathbb{P}^1_{2,2}$ (see for instance [12])

$$\mathbf{F}_{2,2} = \tfrac{1}{\hbar}\left[\tfrac{1}{2}t_0^2 s + \tfrac{1}{4}(t_{(1,1)}^2 + t_{(1,2)}^2)t_0 - \tfrac{1}{96}(t_{(1,1)}^4 + t_{(1,2)}^4) + t_{(1,1)}t_{(1,2)}\mathrm{e}^s z + \tfrac{1}{2}\mathrm{e}^{2s}z^2\right]$$

(where the second index of the cohomology variables refers to one of the two orbifold points) promptly gives

$$A = \frac{1}{4} \qquad B = -\frac{1}{96} \qquad C = 1 \qquad D = \frac{1}{2}$$

Similarly one computes

$$\mathbf{F}_3 = \tfrac{1}{\hbar}\left[\tfrac{1}{3}t_0 t_1 t_2 + \tfrac{1}{18}t_1^3 - \tfrac{1}{36}t_1^2 t_2^2 + \tfrac{1}{648}t_1 t_2^4 - \tfrac{1}{19440}t_3^6 + \left(t_1 + \tfrac{1}{6}t_2^2\right)p_1 + \tfrac{1}{2}t_2 p_2 + \tfrac{1}{3}p_3\right]$$

$$\begin{aligned}\mathbf{F}_4 = \tfrac{1}{\hbar}\Big[&-\tfrac{t_3^8}{4128768} + \tfrac{t_2 t_3^6}{73728} - \tfrac{t_1 t_3^5}{30720} - \tfrac{t_2^2 t_3^4}{3072} + \tfrac{1}{384}t_1 t_2 t_3^3 + \tfrac{1}{384}t_2^3 t_3^2 - \tfrac{1}{64}t_1^2 t_3^2 - \tfrac{1}{32}t_1 t_2^2 t_3 + \tfrac{1}{4}t_0 t_1 t_3 \\ &- \tfrac{t_2^4}{192} + \tfrac{1}{8}t_0 t_2^2 + \tfrac{1}{8}t_1^2 t_2 + \left(\tfrac{t_3^3}{96} + \tfrac{1}{4}t_2 t_3 + t_1\right)p_1 + \left(\tfrac{t_3^2}{8} + \tfrac{t_2}{2}\right)p_2 + \tfrac{1}{3}t_3 p_3 + \tfrac{1}{4}p_4\Big]\end{aligned}$$

$$
\begin{aligned}
\mathbf{F}_5 = \tfrac{1}{\hbar}\Big[& -\tfrac{7t_4^{10}}{8100000000} + \tfrac{7t_3t_4^8}{90000000} - \tfrac{t_2t_4^7}{3150000} - \tfrac{13t_3^2t_4^6}{4500000} + \tfrac{t_1t_4^6}{2250000} + \tfrac{t_3^5}{3000} + \tfrac{11t_2t_3t_4^5}{375000} + \tfrac{7t_3^3t_4^4}{150000} \\
& - \tfrac{t_2^2t_4^4}{7500} - \tfrac{t_1t_3t_4^4}{15000} - \tfrac{1}{150}t_1t_3^3 - \tfrac{t_2t_3^2t_4^3}{1500} + \tfrac{1}{750}t_1t_2t_4^3 + \tfrac{1}{10}t_1t_2^2 - \tfrac{1}{50}t_2^2t_3^2 - \tfrac{3t_3^4t_4^2}{10000} - \tfrac{1}{100}t_1^2t_4^2 \\
& + \tfrac{1}{500}t_1t_3^2t_4^2 + \tfrac{1}{250}t_2^2t_3t_4^2 + \tfrac{1}{10}t_1^2t_3 + \tfrac{1}{5}t_0t_2t_3 - \tfrac{1}{150}t_2^3t_4 + \tfrac{1}{250}t_2t_3^3t_4 + \tfrac{1}{5}t_0t_1t_4 - \tfrac{1}{25}t_1t_2t_3t_4 \\
& + \left(\tfrac{t_4^4}{3000} + \tfrac{1}{50}t_3t_4^2 + \tfrac{t_2t_4}{5} + \tfrac{t_3^2}{10} + t_1\right)p_1 + \left(\tfrac{t_4^3}{75} + \tfrac{t_3t_4}{5} + \tfrac{t_2}{2}\right)p_2 + \left(\tfrac{t_4^2}{10} + \tfrac{t_3}{3}\right)p_3 + \tfrac{1}{4}t_4p_4 + \tfrac{1}{5}p_5\Big]
\end{aligned}
$$

$\triangle$

Now we can proceed with our gluing procedure, which consists, according to Theorem 2.5.3 of [16], in obtaining the GW-potential for the genus g' orbicurve $S_{\alpha_1,\dots,\alpha_a} = \Sigma_{g',(z_1,\alpha_1),\dots,(z_a,\alpha_a)}$, with orbifold points $z_1,\dots,z_a$ and local groups $\mathbb{Z}_{\alpha_1},\dots,\mathbb{Z}_{\alpha_a}$ as

$$
\mathbf{F}_{S_{\alpha_1,\dots,\alpha_a}}(t,s_2) = \log\Big[\exp(\overrightarrow{\mathbf{F}}_{\alpha_1}(t,p))\dots\exp(\overrightarrow{\mathbf{F}}_{\alpha_a}(t,p))\exp(\mathbf{F}_S(t_0,s_2,q))\Big] \tag{3.1}
$$

which gives, according for our general formula for the potential of the orbifold cap, the following expression of the genus g orbifold Gromov-Witten potential of the orbicurve $S_{\alpha_1,\dots,\alpha_a}$

$$
\begin{aligned}
\mathbf{F}_{S_{\alpha_1,\dots,\alpha_a}} = & \frac{1}{\hbar}\frac{t_0^2s_2}{2} - \frac{s_2}{24} \\
& + \sum_{r=1}^{a} \sum_{\substack{i_{(0,r)},\dots,i_{(\alpha_r-1,r)} \\ \sum(\alpha_r-k)i_{(k,r)}=2\alpha_r}} A^r_{i_{(0,r)}\cdots i_{(\alpha_r-1,r)}} t_0^{(i_0,r)} t_{(1,r)}^{i_{(1,r)}} \dots t_{(\alpha_r-1,r)}^{i_{(\alpha_r-1,r)}} \\
& + \sum_{g=0}^{\infty}\sum_{d=0}^{\infty} \sum_{|\mu_1|,\dots,|\mu_a|=d} \mathrm{e}^{ds_2} H_{g,d}^{\Sigma_{g'}}(\mu_1,\dots,\mu_a) \\
& \prod_{r=1}^{a}\left(\sum_{i_{(1,r)},\dots,i_{(\alpha_r-1,r)}} B^r_{i_{(1,r)}\cdots i_{(\alpha_r-1,r)}} t_{(1,r)}^{i_{(1,r)}} \dots t_{(\alpha_r-1,r)}^{i_{(\alpha_r-1,r)}}\right)^{\mu_r} \hbar^{g-1}
\end{aligned}
\tag{3.2}
$$

where, as in Example 3.2.1, the second index in round brackets, as well as the upper index of A and B, specifies one of the a orbifold points.

Notice here that, in the above formula, the sum over the branching configurations $\mu_1,\dots,\mu_a$, where $\mu_k = (\mu_{1,k},\dots,\mu_{s_k,k})$, $k = 1,\dots,a$ is the local branching degree over the k-th point, involves only those terms for which each of the $\mu_{j,k}$ is less or equal than α_k. This phenomenon is very important and sometimes it leaves only a finite number of nonzero terms in genus 0, making the rational GW-potential $\mathbf{f}_{S_{\alpha_1,\dots,\alpha_a}} := \left(\hbar\,\mathbf{F}_{S_{\alpha_1,\dots,\alpha_a}}\right)\big|_{\hbar=0}$ a polynomial in the variables $t_0, t_{(1,1)},\dots,t_{(\alpha_1-1,1)},\dots,t_{(1,a)},\dots,t_{(\alpha_a-1,a)},\mathrm{e}^{s_2}$. Our next task will consist in classifying exactly these cases, at least for $g'=0$.

3.3 Polynomial $\mathbb{P}^1$-orbifolds

In this section we study the rational orbifold Gromov-Witten theory of those genus 0 orbicurves which give rise to a polynomial quantum cohomology, i.e. those whose associated genus 0 Gromov-Witten potential is polynomial in the variables $t_0, t_{(1,1)},\dots,t_{(\alpha_1-1,1)},\dots,t_{(1,a)},\dots,t_{(\alpha_a-1,a)},\mathrm{e}^s$. We will call such orbifold Riemann surfaces *polynomial* $\mathbb{P}^1$*-orbifolds.*

Lemma 3.3.1. *The only polynomial* $\mathbb{P}^1$*-orbifolds are those* $\mathbb{P}^1_{(z_1,\alpha_1),\dots,(z_a,\alpha_a)}$ *such that:*

1) $a = 0, 1, 2$, for any (α_1, α_2):

$$\mathbb{P}^1_{\alpha_1,\alpha_2}$$

2) $a = 3$, $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, l-2)$, $l \geq 4$:

$$\mathbb{P}^1_{2,2,l-2}$$

3) $a = 3$, $(\alpha_1, \alpha_2, \alpha_3) = (2, 3, 3), (2, 3, 4), (2, 3, 5)$:

$$\mathbb{P}^1_{2,3,5} \qquad \mathbb{P}^1_{2,3,4} \qquad \mathbb{P}^1_{2,3,5}$$

Proof. A necessary and sufficient condition for the finiteness of the potential is obtained by using Riemann-Hurwitz theorem. Indeed, validity of the Riemann-Hurwitz relation

$$\sum_{k=1}^{a}\sum_{i=1}^{s_k}(\mu_{i,k} - 1) = 2d - 2$$

is a necessary condition for existence of a degree d covering with branching profile given by $\mu_1, \dots, \mu_a$, with $\mu_k = (\mu_{1,k}, \dots, \mu_{s_k,k})$. As we already noticed, the branched coverings appearing in our formula for the potential of $\mathbb{P}^1_{(z_1,\alpha_1),\dots,(z_a,\alpha_a)}$ have a branching profile with local branching degrees $\mu_k = (\mu_{1,k}, \dots, \mu_{s_k,k})$, $k = 1, \dots, a$ such that $\mu_{i,k} \leq \alpha_k$, $i = 1, \dots, s_k$. Denoting by $n_{j,k}$ the number of occurrences of j in the set $\{\mu_{1,k}, \dots, \mu_{s_k,k}\}$, we can then rewrite the Riemann-Hurwitz relation as

$$\sum_{k=1}^{a}\sum_{j=1}^{\alpha_k}(j-1)n_{j,k} = 2d - 2$$

We are interested in singling out the situations where there are only a finite number of positive integer solutions $n_{j,k}$ to this equation, together with the constant degree conditions $d = \sum_{j=1}^{\alpha_k} j\, n_{j,k}$, $k = 1 \dots, a$. It is immediately clear (consider the branching configurations where $n_{1,1} = 4$, $n_{2,1} = 2r - 2$, $n_{2,2} = n_{2,3} = n_{2,4} = 2r$ are the only nonzero coefficients) that this never happens when $a \geq 4$, while it is always the case when $a \leq 2$. Finally, when $a = 3$ we distinguish between the following cases. For $\alpha_1 = \alpha_2 = 2$, to ensure positiveness of $n_{1,1}, n_{2,1}, n_{1,2}$ we must have

$$(\alpha_3 + 2)n_{\alpha_3,3} + (\alpha_3 + 1)n_{\alpha_3-1,3} + \ldots + 3n_{1,3} - 4 \leq 2n_{2,2}$$

$$(\alpha_3 + 1)n_{\alpha_3,3} + \alpha_3 n_{\alpha_3-1,3} + \ldots + 2n_{1,3} - 2 \geq n_{2,2}$$

$$\alpha_3 n_{\alpha_3,3} + (\alpha_3 - 1)n_{\alpha_3-1,3} + \ldots + n_{1,3} \geq 2n_{2,2}$$

whence $2(n_{\alpha_3,3} + \ldots + n_{1,3}) \leq 4$, so we get a finite number of integer solutions for any α_3. For $\alpha_1 = 2$ and $\alpha_2 = 3$ we have similarly

$$(\alpha_3 + 2)n_{\alpha_3,3} + (\alpha_3 + 1)n_{\alpha_3-1,3} + \ldots + 3n_{1,3} - 4 \leq 2n_{2,2} + n_{3,1}$$

$$(\alpha_3 + 1)n_{\alpha_3,3} + \alpha_3 n_{\alpha_3-1,3} + \ldots + 2n_{1,3} - 2 \geq n_{2,2} + 2n_{3,1}$$

$$\alpha_3 n_{\alpha_3,3} + (\alpha_3 - 1)n_{\alpha_3-1,3} + \ldots + n_{1,3} \geq 2n_{2,2}$$

whence $(6-\alpha_3)n_{\alpha_3,3}+\ldots+5n_{1,3}\leq 12$, so the solutions are in a finite number if $\alpha_3\leq 5$. Vice versa, if $\alpha_3\geq 6$, we have infinite sequences of solutions like $n_{2,2}=3r$, $n_{3,1}=2r-2$, $n_{6,3}=r$ and $n_{j,k}=0$ otherwise.
Other values of α_1,α_2 always give infinite solutions.

Notice that the Riemann-Hurwitz relation is not a sufficient condition and the problem of determining all the admissible (satisfying Riemann-Hurwitz) covering configurations that are not actually geometrically realizable is classical and still open (see e.g. [30] for a review about this challenging topic). Nonetheless the non-realizable cases are quite exceptional and (as shown in [30]) by no means influence infiniteness of the number of actual coverings. This ensures that we have actually found all the polynomial $\mathbb{P}^1$-orbifolds. $\square$

In the above classification, case 1) was studied in [25], where the quantum cohomology of $\mathbb{P}^1_{\alpha_1,\alpha_2}$ was shown to be isomorphic, as Frobenius manifold, to the space $M(A_l,k)$ of Fourier polynomials invariant with respect to the extended affine Weyl group of the root system A_l, with the choice of the k-th root and with $l=\alpha_1+\alpha_2-1$ and $k=\alpha_1$. We will not remind here the general procedure (see [12]) to construct the Frobenius structure on the space $M(R,k)$. However we recall that polynomial Frobenius manifolds of dimension $l+1$ can be associated with root systems of type D_l and E_l, but in this case the choice of the k-th root is forced to be at the bifurcation of the Dynkin diagram. Moreover, the relevant Frobenius potential happens to be homogeneous with respect to a specific grading for the coordinates, and this grading coincides (up to an irrelevant factor) with the one we defined above once we compare the manifolds $M(D_l,l-2)$ with $QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,2,l-2})$ and $M(E_l,4)$ with $QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,3,l-3})$.

With the idea to investigate this correspondence, we plan to compute explicitly the genus 0 Gromov-Witten potential $\mathbf{f}_{\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}}$ of the above polynomial $\mathbb{P}^1$-orbifolds. This can be done easily (although computations can get quite cumbersome) by using equation (3.2) above, together with Riemann-Hurwitz relation, as in the proof of the above Lemma, to predict all the (a priori) non-zero Hurwitz numbers appearing as coefficients in our formula. This leaves us with a closed expression for $\mathbf{f}_{\mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}}$ which still contains a finite number of Hurwitz coefficients. These can then be determined by imposing WDVV equations. Without reporting the details of these computations, we give some examples of the results of this construction.

Example 3.3.2.

$$\begin{aligned}\mathbf{f}_{\mathbb{P}^1_{2,2,2}} &= \tfrac{1}{4}e^{4t_4}z^4+\tfrac{1}{2}e^{2t_4}\left(t_1^2+t_2^2+t_3^2\right)z^2+e^{t_4}t_1t_2t_3z+\tfrac{1}{4}t_0\left(t_1^2+t_2^2+t_3^2\right)+\tfrac{1}{96}\left(t_1^4+t_2^4+t_3^4\right)\\ &\quad+\tfrac{1}{2}t_0^2t_4\end{aligned}$$

$$\begin{aligned}\mathbf{f}_{\mathbb{P}^1_{2,2,3}} &= \tfrac{1}{6}e^{6t_5}z^6+\tfrac{1}{4}e^{4t_5}t_4^2z^4+e^{3t_5}t_1t_2z^3+\tfrac{1}{2}e^{2t_5}\left(\tfrac{t_4^2}{6}+t_3\right)^2z^2+\tfrac{1}{2}e^{2t_5}\left(t_1^2+t_2^2\right)t_4z^2\\ &\quad+e^{t_5}t_1t_2\left(\tfrac{t_4^2}{6}+t_3\right)z-\tfrac{t_4^6}{19440}+\tfrac{1}{648}t_3t_4^4+\tfrac{t_3^3}{18}-\tfrac{1}{36}t_3^2t_4^2+\tfrac{1}{4}t_0\left(t_1^2+t_2^2\right)-\tfrac{1}{96}\left(t_1^4+t_2^4\right)\\ &\quad+\tfrac{1}{3}t_0t_3t_4+\tfrac{1}{2}t_0^2t_5\end{aligned}$$

$$\begin{aligned}\mathbf{f}_{\mathbb{P}^1_{2,2,4}} &= \tfrac{1}{8}e^{8t_6}z^8 + \tfrac{1}{6}e^{6t_6}t_5^2z^6 + e^{4t_6}\left(\tfrac{t_5^2}{8}+\tfrac{t_4}{2}\right)^2 z^4 + \tfrac{1}{2}e^{4t_6}\left(t_1^2+t_2^2\right)z^4 + e^{3t_6}t_1t_2t_5z^3 \\ &+\tfrac{1}{2}e^{2t_6}\left(\tfrac{t_5^3}{96}+\tfrac{t_4t_5}{4}+t_3\right)^2 z^2 + e^{2t_6}\left(t_1^2+t_2^2\right)\left(\tfrac{t_5^2}{8}+\tfrac{t_4}{2}\right)z^2 + e^{t_6}t_1t_2\left(\tfrac{t_5^3}{96}+\tfrac{t_4t_5}{4}+t_3\right)z \\ &-\tfrac{t_5^8}{4128768}+\tfrac{t_4t_5^6}{73728}-\tfrac{t_3t_5^5}{30720}-\tfrac{t_4^4}{192}-\tfrac{t_4^2t_5^4}{3072}+\tfrac{1}{384}t_3t_4t_5^3+\tfrac{1}{8}t_0t_4^2+\tfrac{1}{384}t_4^3t_5^2-\tfrac{1}{64}t_3^2t_5^2 \\ &+\tfrac{1}{4}t_0\left(t_1^2+t_2^2\right)+\tfrac{1}{96}\left(-t_1^4-t_2^4\right)+\tfrac{1}{8}t_3^2t_4-\tfrac{1}{32}t_3t_4^2t_5+\tfrac{1}{4}t_0t_3t_5+\tfrac{1}{2}t_0^2t_6\end{aligned}$$

$$\begin{aligned}\mathbf{f}_{\mathbb{P}^1_{2,3,3}} &= -\tfrac{t_5^4}{96}+\tfrac{1}{3}e^{3t_6}t_5^3+\tfrac{1}{2}e^{6t_6}t_5^2+\tfrac{1}{4}t_0t_5^2+\tfrac{1}{2}e^{2t_6}t_2t_4t_5^2+e^{5t_6}t_2t_4t_5+e^{t_6}\left(\tfrac{t_2^2}{6}+t_1\right)\left(\tfrac{t_4^2}{6}+t_3\right)t_5 \\ &+e^{3t_6}\left(t_2\left(\tfrac{t_2^2}{6}+t_1\right)+t_4\left(\tfrac{t_4^2}{6}+t_3\right)\right)t_5+\tfrac{e^{12t_6}}{12}+\tfrac{1}{4}e^{4t_6}t_2^2t_4^2+\tfrac{1}{18}\left(t_1^3+t_3^3\right)+\tfrac{1}{2}e^{8t_6}t_2t_4 \\ &+\tfrac{1}{3}t_0\left(t_1t_2+t_3t_4\right)+e^{4t_6}\left(\tfrac{t_2^2}{6}+t_1\right)\left(\tfrac{t_4^2}{6}+t_3\right)+\tfrac{1}{36}\left(-t_1^2t_2^2-t_3^2t_4^2\right)+\tfrac{1}{6}e^{6t_6}\left(t_2^3+t_4^3\right) \\ &+\tfrac{1}{648}\left(t_1t_2^4+t_3t_4^4\right)+\tfrac{-t_2^6-t_4^6}{19440}+\tfrac{1}{2}e^{2t_6}\left(t_4\left(\tfrac{t_2^2}{6}+t_1\right)^2+t_2\left(\tfrac{t_4^2}{6}+t_3\right)^2\right)+\tfrac{1}{2}t_0^2t_6\end{aligned}$$

△

All the above potentials turn out to coincide with know examples from extended affine Weyl groups and, in particular, one can directly verify the isomorphism of Frobenius manifolds

$$M(E_l,4)\cong QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,3,l-3})\qquad l=6,7,8$$

together with instances of the other isomorphisms appearing in theorem 4.0.3. However a complete proof of our mirror result will require a less computational approach, which is what we plan for the next chapter.

3.4 Symplectic Field Theory of Seifert fibrations

What we are going to explain now is a generalization of Proposition 2.9.2 of [16] to the case of oriented Seifert fibrations over orbifold Riemann surfaces. For simplicity we will consider just the case of an oriented Seifert S^1-orbibundle $\pi: V\to P$ over a $\mathbb{P}^1$-orbifold $P=\mathbb{P}^1_{(z_1,\alpha_1),\dots,(z_a,\alpha_a)}$ with uniformizing systems at z_i given by $z\mapsto z^{\alpha_i}$, with Seifert invariant $(c,\beta_1,\dots,\beta_a)$ (here c is the first Chern class of the orbibundle V and $b=c-\sum\frac{\beta_i}{\alpha_i}$ is the first Chern class of its de-singularization $|V|$, as in [8]). This is actually a sort of mildly singular Hamiltonian structure of fibration type, since the manifold V itself is still smooth, but the Reeb orbit space is a symplectic orbifold with $\mathbb{Z}_n$-singularities. However it satisfies the *Morse-Bott* condition of [4]. Moreover we stick to the rational ($g=0$) SFT. Here we use Chen and Ruan's theory and terminology of orbifolds [8],[9].

The cylindrical cobordism $V\times\mathbb{R}$ can be seen as the total space of the complex line orbibundle L associated with $V\to P$, which is actually holomorphic for a proper choice of the almost complex structure J, and with the zero section (containing the singular locus of L) removed. Following [16] we want to project a SFT-curve $u:\mathbb{P}^1-\{x_1,\dots,x_s\}\to L$ along π to obtain a stable orbifold GW-curve $\bar u$ to P. This is trivial outside $\pi^{-1}(z_i)$, while the following Lemma ensures that the domain curve $\mathbb{P}^1-\{x_1,\dots,x_s\}$ can be given in a unique way an orbifold structure at the punctures $\{x_1,\dots,x_s\}$ in such a way that we get a genuine stable GW-map to P.

Lemma 3.4.1. *Let $u : \Sigma \to \Sigma'_{(z_1,\alpha_1),\dots,(z_a,\alpha_a)}$ be a non-constant holomorphic map between a Riemann surface and a complex orbicurve. Then there is a unique orbifold structure on Σ (which we denote Σ_u) and a unique germ of C^∞-lift $\tilde{u}$ of u (which is regular) such that the group homomorphism at each point is injective.*

Proof. There are finitely many $x_i \in \Sigma$ such that $u(x_i) = z_i$. An orbifold structure at x_i and the corresponding C^∞-lift of u fit into the diagram

$$\begin{tikzcd} D \arrow[r, "z^q"] \arrow[d, "z^p"'] & D \arrow[d, "z^{\alpha_i}"] \\ D_{x_i} \arrow[r, "z^n"] & D_{z_i} \end{tikzcd}$$

where n is assigned with the map u. So we have $pn = q\alpha_i$. Now, if p and q have some common factor, then the group homomorphism is not injective (the common factor $k \in \mathbb{Z}_p$ is sent to $0 \in \mathbb{Z}_{\alpha_i}$), then p and q are uniquely determined by n and α_i thanks to the injectivity condition. The map is obviously regular by construction. □

This way we get a fibration between the relevant moduli spaces of holomorphic maps

$$\mathrm{pr} : \mathcal{M}^A_{0,r,s}(L) \to \mathcal{M}^A_{0,r+s}(P)$$

where $A \in H_2(P,\mathbb{Z}) = \mathbb{Z}$. Using the orbicurve version of the correspondence between effective divisors and line bundles on P, one is able to identify the fiber $\mathrm{pr}^{-1}(\bar{u})$. As in the smooth case, u can be reconstructed from $\bar{u}$ by assigning a meromorphic section of $\bar{u}^*(L)$ which is determined, up to an S^1-symmetry, by its divisor of poles and zeros, hence by s of the $r+s$ marked points on $\mathbb{P}^1_{\bar{u}}$ and a sequence of integers $(k_1,\dots,k_s)$ such that

$$\sum_{i=1}^{s} \frac{k_i}{m_i} = c_1(\bar{u}^*(L)) = \deg(\bar{u})\, c_1(L) = \deg(\bar{u}) \left(c_1(|L|) + \sum_{j=1}^{a} \frac{\beta_i}{\alpha_i} \right),$$

where m_i is the orbifold multiplicity of the i-th of the s points of the divisor on $\mathbb{P}^1_{\bar{u}}$ and $|L|$ is the de-singularization of L (see e.g. [8]).

Let now $\Delta_1,\dots,\Delta_b$ be a basis of $H^*_{\mathrm{orb}}(P)$ such that the system of forms $\tilde{\Delta}_j := \pi^*(\Delta_j)$, $j = 1,\dots,c < b$ generate $\pi^*(H^*_{\mathrm{orb}}(P)) = \pi^*(H^*(P)) \subset H^*(V)$, and the forms $\tilde{\Theta}_1,\dots,\tilde{\Theta}_d$ complete it to a basis of $H^*(V)$. Notice also that $H_2(V,\mathbb{Z}) \cong H^1(V,\mathbb{Z}) = 0$ (if V is not trivial).
Extending the proof in [4][16] to obtain the following generalization is now a trivial matter.

Proposition 3.4.2. *Let $\mathbf{f}_P(\sum t_i\Delta_i, z)$ be the genus 0 orbifold Gromov-Witten potential of P and $\mathbf{h}_V(\sum t_i\tilde{\Delta}_i + \sum \tau_j\tilde{\Theta}_j, q, p)$ the rational SFT potential of V (as a framed Hamiltonian structure of fibration type). Let*

$$\mathbf{h}^j_V(t,q,p) = \frac{\partial \mathbf{h}_V}{\partial \tau_j} \left(\sum_1^c t_i\tilde{\Delta}_i + \tau_j\tilde{\Theta}_j, q, p \right)\Bigg|_{\tau_j=0}$$

$$\mathbf{f}^j_P(t;z) = \frac{\partial \mathbf{f}_P}{\partial s} \left(\sum_1^b t_i\Delta_i + s\pi_*\tilde{\Theta}_j, z \right)\Bigg|_{s=0}$$

for $j = 1, \ldots, d$ and where π_* denotes integration along the fibers of V. Then we have

$$\mathbf{h}_V^j(t,q,p) = \frac{1}{2\pi\alpha_1\ldots\alpha_a}\int_0^{2\pi\alpha_1\ldots\alpha_a} \mathbf{f}_P^j(t_1{+}u_1(x),\ldots,t_b{+}u_b(x),u_{b+1}(x),\ldots,u_c(x);e^{-\mathrm{i}c_1(V)x})\mathrm{d}x$$

where

$$u_n(x) := \sum_{k=1}^{\infty}\left(q_{k\iota_n,n}\,e^{-\mathrm{i}\,k\iota_n x} + p_{k\iota_n,n}\,e^{\mathrm{i}\,k\iota_n x}\right) \qquad n = 1,\ldots,b$$

and ι_n denotes the degree shifting (in the sense of orbifold cohomology, see [8][9]) of the twisted sector of P where the form Δ_n is defined.

More explicitly, up to degree shifting and just as a vector space, the orbifold cohomology of P is the singular cohomology (with $\mathbb{R}$ coefficients) of the inertia orbifold of P, hence just the disjoint union of a sphere S^2 and $[(\alpha_1-1)+\ldots+(\alpha_a-1)]$ points, labeled by a couple of indices $(i=1,\ldots,a; l=1,\ldots,\alpha_i-1)$. Let us choose the $(\alpha_1+\ldots+\alpha_a-a+2)$ elements of the basis of $H^*_{\mathrm{orb}}(P)$ in the following way: $\Delta_1 = 1, \Delta_2 = [\omega] \in H^*(S^2)$ and $\Delta_{(i,l)} = 1 \in H^0(\mathrm{pt}_{(i,l)})$. Then $\iota_1 = \iota_2 = 0$ and $\iota_{(i,l)} = \frac{l}{\alpha_i}$

Of course, as in the smooth case of [16], the Poisson structure and grading on the graded Poisson algebra $\mathfrak{U}$, relevant for rational SFT and where $\mathbf{h}_V$ is defined, is given again and in the same way by the orbifold Poincaré pairing and grading in $H^*_{\mathrm{orb}}(P)$ (see [8]).

Thanks to this generalization we can use the Gromov-Witten potentials of $\mathbb{P}^1$-orbifolds computed in the previous sections to obtain explicit expressions for the SFT-Hamiltonians of some interesting contact 3-manifolds, which are Seifert fibrations over these orbifolds. The A-case gives the general Lens spaces, the D-cases are Prism manifolds, while among the exceptional E-cases one can find quotients of the Poincaré sphere and more exotic manifolds.

Chapter 4

The mirror model to $QH^*_{\mathrm{orb}}(\mathbb{P}^1_{p,q,r})$

In this chapter we construct a family of Frobenius manifolds on the space $M_{p,q,r}$ of polynomials in three complex variables of the form

$$F(x,y,z) = -xyz + P_1(x) + P_2(y) + P_3(z)$$

with given degrees p, q, r in x, y, z respectively and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

The cases $M_{p,q,1}$ are easily seen to coincide with the spaces of Laurent polynomials of [12], which have been shown ([25]) to be isomorphic as Frobenius manifolds to $QH^*_{\mathrm{orb}}(\mathbb{P}^1_{k,l-k+1})$. This motivates our mirror theorem

Theorem 4.0.3. *Let $M_{p,q,r}$ be the space of tri-polynomials. Then we have the following isomorphisms of Frobenius manifolds:*

$$M_{p,q,r} \cong QH^*_{\mathrm{orb}}(\mathbb{P}^1_{p,q,r})$$

An Homological Mirror Symmetry version of this result was conjectured by Takahashi in [35], where tri-polynomials also appear. Very recently prof. Takahashi communicated he found a proof of this Homological Mirror Symmetry result which should hold for general p, q, r. Equivalence of Homological Mirror Symmetry with isomorphism of corresponding Frobenius manifolds is generally believed to be true, but far from being proven. This one can be seen as an instance where both the versions of Mirror Symmetry hold.

Notice, moreover, that $M_{2,2,r}$ can in turn be proven to be isomorphic to $M(D_{r+2}, r)$, the Frobenius manifold associated to the extended affine Weyl group $\tilde{D}_r$ (see [12]). This can be promptly done by using Dubrovin and Zhang's reconstruction Theorem 2.1 of [12], which states that there is only one polynomial Frobenius structure with the unity vector field, Euler vector field and intersection pairing of $M(D_{r+2}, r)$. It is then sufficient to check that $M_{p,q,r}$ has precisely the same e, E and intersection metric, the last one given by

$$(w_1, w_2)^{\sim} = i_E(w_1 \bullet w_2)$$

where we identify 1-forms and vectors using the other metric of the Frobenius manifold.

In fact polynomiality follows from the isomorphism $M_{2,2,r} \simeq QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,2,r})$ and our classification theorem of $\mathbb{P}^1$-orbifolds.

The exceptional cases E_6, E_7, E_8 are even more easily dealt with explicitly computing and comparing the Frobenius potentials. Hence, let $M(R,k)$ be the Frobenius manifold associated to the extended affine Weyl group of the root system R with the choice of the k-th root, according to [12]. Then we have the following isomorphisms of Frobenius manifolds:

$$\begin{aligned} M(A_l,k) &\cong QH^*_{\mathrm{orb}}(\mathbb{P}^1_{k,l-k+1}) && \text{see [25]} \\ M(D_l,l-2) &\cong QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,2,l-2}) && \\ M(E_l,4) &\cong QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,3,l-3}) && l=6,7,8 \end{aligned}$$

4.1 Space of tri-polynomials

We will denote by $M_{p,q,r}$ the space of polynomials (we will refer to them as tri-polynomials) of the form

$$F(x,y,z) = -xyz + P_1(x) + P_2(y) + P_3(z)$$

where

$$P_1(x) = \sum_{k=1}^{p} a_k x^k \qquad P_2(y) = \sum_{k=1}^{q} b_k y^k \qquad P_3(z) = \sum_{k=0}^{r} c_k (\mathrm{e}^d z)^k$$

and normalized by $a_p = b_q = c_r = 1$. This means that $M_{p,q,r} \cong \mathbb{C}^{p+q+r-2} \times \mathbb{C}^*$. It is an easy exercise with generators and relations to show that the condition on p,q,r such that the local algebra $\mathbb{C}[x,y,z]/J_F$, where J_F is the Jacobian ideal of F (i.e. $J_F = (\partial_x F, \partial_y F, \partial_z F)$), is isomorphic as a $\mathbb{C}$-module to the tangent space $T_F M_{p,q,r}$ is $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. We will assume this condition is always verified in what follows. Moreover, up tp permutation of x,y,z, we will just need to consider the cases

(A) $(p,q,r) = (p,q,1)$, $\quad p,q = 1,2,\dots$

(D) $(p,q,r) = (2,2,r)$, $\quad r = 2,3,\dots$

(E) $(p,q,r) = (2,3,r)$, $\quad r = 3,4,5$

We are going to define a Frobenius manifold structure on $M_{p,q,r}$. Similarly to the case of Laurent polynomials (see e.g. [10],[25]), the isomorphism of each tangent space $T_F M_{p,q,r}$ with the local algebra $\mathbb{C}[x,y,z]/J_F$, defines a commutative associative algebra structure on $T_F M_{p,q,r}$, with unity $e = \partial/\partial c_0$.

We then equip each tangent space with the residue pairing

$$(\partial,\partial')_F = \underset{\substack{F(x,y,z)\neq\infty \\ \partial_x F = \partial_y F = \partial_z F = 0}}{\mathrm{res}} \frac{\partial(F)\,\partial'(F)}{\partial_x F\,\partial_y F\,\partial_z F}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \tag{4.1}$$

and assign the following grading to the variables

$$\deg a_i = -2 + \frac{2i}{p} \qquad i = 1,\dots,p-1$$

$$\deg b_j = -2 + \frac{2j}{q} \qquad j = 1, \ldots, q-1$$
$$\deg c_k = -2 + \frac{2k}{r} \qquad k = 0, \ldots, r-1$$
$$\deg d = 0$$

which gives the following Euler vector field

$$E = \sum_{i=1}^{p-1}(1-\frac{i}{p})a_i\frac{\partial}{\partial a_i} + \sum_{j=1}^{q-1}(1-\frac{j}{q})b_j\frac{\partial}{\partial b_j} + \sum_{k=0}^{r-1}(1-\frac{k}{r})c_k\frac{\partial}{\partial c_k} + (-1+\frac{1}{p}+\frac{1}{q}+\frac{1}{r})\frac{\partial}{\partial d}$$

Notice that, as usual in singularity theory, E corresponds to $F(x,y,z)$ via the isomorphism $TM_{p,q,r} \simeq \mathbb{C}[x,y,z]/J_F$.

All these structures are easily verified to be compatible, hence it only remains to prove flatness of the metric (4.1) and potentiality (i.e. the symmetry of $\nabla_X g(Y, Z \circ W)$ in X, Y, Z, W). The first requirement is achieved by exhibiting a system of flat coordinates. We start with the cases A and D. Let

$$\alpha_i = \operatorname*{res}_{x=\infty} \frac{p}{i}\frac{F(x,0,0)^{1-\frac{i}{p}}}{x}\mathrm{d}x \qquad i = 1, \ldots, p-1$$

$$\beta_j = \operatorname*{res}_{y=\infty} \frac{q}{j}\frac{F(0,y,0)^{1-\frac{j}{q}}}{y}\mathrm{d}y \qquad j = 1, \ldots, q-1$$

$$\gamma_k = \operatorname*{res}_{z=\infty} \frac{r}{k}\frac{F(0,0,z)^{1-\frac{k}{r}}}{\sqrt{z^2-4}}\mathrm{d}z \qquad k = 0, \ldots, r-1$$

where, as specified, either $(p,q,r) = (p,q,1)$ or $(p,q,r) = (2,2,r)$.

Lemma 4.1.1. *In the cases A and D, the functions $\alpha_i, \beta_j, \gamma_k : M_{p,q,r} \to \mathbb{C}$, together with $d : M_{p,q,r} \to \mathbb{C}$, form a system of flat coordinates for the metric (4.1), such that the only nonzero pairings of basis vectors are given by*

$$(\partial_{\alpha_{i_1}}, \partial_{\alpha_{i_2}}) = \frac{1}{p}\,\delta_{i_1+i_2,p}$$

$$(\partial_{\beta_{j_1}}, \partial_{\beta_{j_2}}) = \frac{1}{q}\,\delta_{j_1+j_2,q}$$

$$(\partial_{\gamma_{k_1}}, \partial_{\gamma_{k_2}}) = \frac{1}{r}\,\delta_{k_1+k_2,r}$$

$$(\partial_{\gamma_0}, \partial_d) = 1$$

with indices ranging as above.

Proof. In the A case the formulas for α_i and β_j coincide with the ones given in [12] for the space of Laurent polynomials, hence only the D case is left to prove.

From the above formulae we see that $\alpha_1 = a_1$ and $\beta_1 = b_1$. We now prove that $(\partial_{\gamma_{k_1}}, \partial_{\gamma_{k_2}}) = \frac{1}{r}\,\delta_{k_1+k_2,r}$. Consider the solutions, with respect to z, to $F(x,y,z) = \lambda$, for λ and z near infinity, and denote it by $z = z(\lambda; x, y)$. Using the chain rule we get

$\partial_{\gamma_k}F = -(\partial_z F)(\partial_{\gamma_k} z)$. Moreover, in the D case, the residues for the matrix elements of the metric with respect to x and y localize at the poles

$$\begin{cases} \partial_x F(x,y,z) = 0 \\ \partial_y F(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} x_p = \frac{b_1 z + 2a_1}{z^2-4} \\ y_p = \frac{a_1 z + 2b_1}{z^2-4} \end{cases}$$

Hence, applying the residue formula in many variables gives

$$\begin{aligned}(\partial_{\gamma_{k_1}}, \partial_{\gamma_{k_2}}) &= -\operatorname*{res}_{\lambda=\infty} \frac{(\partial_{\gamma_{k_1}} z)(\partial_{\gamma_{k_2}} z)}{z^2-4}\, \mathrm{d}\lambda \\ &= -\operatorname*{res}_{\lambda=\infty} (\partial_{\gamma_{k_1}} \log(z+\sqrt{z^2-4}))(\partial_{\gamma_{k_2}} \log(z+\sqrt{z^2-4}))\, \mathrm{d}\lambda\end{aligned}$$

and the coordinates (x_p, y_p) make no appearance in the formula, thanks to the fact that the coordinates γ_i only depend on c_i and d.

The result then follows if we consider the expansion, for λ near infinity, of $\log(z+\sqrt{z^2-4}))$ where the coefficient of $\lambda^{k/r}$ is given by γ_k. Indeed, let the coefficients γ_k be defined by

$$\log(z+\sqrt{z^2-4}) = \frac{1}{r}\left[\log\lambda - \gamma_{r-1}\lambda^{-\frac{1}{r}} - \ldots - \gamma_1 \lambda^{-\frac{r-1}{r}} - \gamma_0 \lambda^{-1}\right] + O(\lambda^{-1-\frac{1}{r}})$$

Then one has

$$\operatorname*{res}_{z=\infty} \frac{r}{k} \frac{\lambda^{1-\frac{k}{r}}}{\sqrt{z^2-4}} \mathrm{d}z = \operatorname*{res}_{z=\infty} \frac{r}{k} \lambda^{1-\frac{k}{r}} \left(\frac{\partial}{\partial \lambda^{\frac{1}{r}}} \left(\log \frac{z+\sqrt{z^2-4}}{\lambda^{\frac{1}{r}}} \right) + \frac{1}{\lambda^{\frac{1}{r}}} \right) \mathrm{d}\lambda^{\frac{1}{r}} = \gamma_k$$

Expressions for $(\partial_{\alpha_1}, \partial_{\alpha_1})$, $(\partial_{\beta_1}, \partial_{\beta_1})$ and $(\partial_{\gamma_0}, \partial_d)$ are proved analogously. This also shows that these are the only nonzero entries. □

The E case is to be dealt with similarly, but it needs more care. First one finds the poles with respect to x and y in the integrand of (4.1):

$$\begin{cases} x_p = \frac{1}{24}\left(z^3 - 12a_1 - 4b_2 z \pm z\sqrt{z^4 - 8b_2 z^2 - 24a_1 z + 16b_2^2 - 48b_1}\right) \\ y_p = \frac{1}{12}\left(z^2 - 4b_2 \pm \sqrt{z^4 - 8b_2 z^2 - 24 a_1 z + 16b_2^2 - 48b_1}\right) \end{cases}$$

(we will use subscripts $p = p_1$ for the choice of plus and $p = p_2$ for the choice of minus in the coordinates above) and, applying once more the residue formula, for any two coordinates t_1 and t_2 one gets

$$(\partial_{t_1}, \partial_{t_2}) = \operatorname*{res}_{\lambda=\infty} \left(\frac{[(\partial_{t_1} z)(\partial_{t_2} z)]_{(x=x_{p_1}, y=y_{p_1})} - [(\partial_{t_1} z)(\partial_{t_2} z)]_{(x=x_{p_2}, y=y_{p_2})}}{\sqrt{z^4 - 8b_2 z^2 - 24a_1 z + 16b_2^2 - 48b_1}} \right) \mathrm{d}\lambda$$

At this point the esasiest thing is probably just computing the flat coordinates from the topological side (recall that we are able to find the explicit expression of the Frobenius potential for $\mathbb{P}^1_{2,3,r}$, $r = 3,4,5$) and plug them into the above formula to check that they are flat coordinates for the metric (4.1) too. For instance, for the E_6 case (see Example 3.3.2), one gets:

$$\alpha_1 = a_1 - 8\mathrm{e}^{3d}$$

$$\beta_1 = b_1 - \frac{b_2^2}{6} + 3c_2 \mathrm{e}^{2d}\,, \qquad \beta_2 = b_2$$

$$\gamma_0 = c_0 + 2a_2b_2\mathrm{e}^{2d} + 6a_1\mathrm{e}^{3d} - 18\mathrm{e}^{6d}$$

$$\gamma_1 = c_1 - \frac{c_2^2}{6} + 3b_2\mathrm{e}^{2d}\ , \qquad \gamma_2 = b_2$$

As for potentiality, one can use the same technique as in Hertling's book on singularities and Frobenius structures [20]. Recall that an F-manifold structure $(M, \circ, e)$ on a complex manifold M is given by a commutative and associative multiplication $\circ$ on TM and a unity vector field e such that $\mathrm{Lie}_{X\circ Y}(\circ) = X \circ \mathrm{Lie}_Y(\circ) + Y \circ \mathrm{Lie}_X(\circ)$. Indeed we use the following result from [20].

Theorem 4.1.2 ([20]). *Let $(M, \circ, e, g)$ be a manifold with a commutative and associative multiplication $\circ$ on TM, a unity vector field e and a metric g which is multiplication invariant. Denote by ϵ the 1-form (coidentity) $g(\cdot, e)$. Then the following are equivalent:*

i) M carries a structure of an F-manifold and ϵ is closed

ii) $\nabla_{\cdot}(\cdot, \cdot \circ \cdot)$ is symmetric in all four arguments

In our case $e = \partial_{c_0} = \partial_{\gamma_0}$, so $\mathrm{d}\epsilon = 0$ follows from flatness of e. The F-manifold condition can be deduced from the following theorem.

Theorem 4.1.3 ([20]). *Let $\pi_Z : Z \twoheadrightarrow M$ be a submersion between the manifolds Z and M, with $\dim M = n$. Let $C \subset Z$ be a an n-dimensional reduced subvariety such that the restriction $\pi_C : C \twoheadrightarrow M$ is finite. Finally, let α_Z be a 1-form on Z such that, for any local lift $\tilde{X} \in \Gamma(TZ)$ of the zero vector field on M, $\alpha_Z(\tilde{X})|_C = 0$.*

Then the map

$$\Gamma(TM) \to (\pi_C)_*\mathcal{O}_C \qquad X \mapsto \alpha_Z(\tilde{X})|_C$$

is well defined, and provides M with a structure of F-manifold (with generically semi-simple multiplication) if and only if it is an isomorphism and $\alpha_Z|_{C_{\mathrm{reg}}}$ is exact.

In our case $Z = \mathbb{C}^3 \times M_{p,q,r}$, $M = M_{p,q,r}$, $C = \{\mathrm{d}_{\mathbb{C}^3}F(x, y, z) = 0\}$ so that $(\pi_C)_*\mathcal{O}_C = (\mathbb{C}[x, y, z] \otimes \mathcal{O}_M)/J_F$, and $\alpha_Z = \mathrm{d}_Z F$, so the hypothesis of the above theorem are easily verified.

Notice that the spaces of tri-polynomials given by $M_{p,q,1}$ are trivially isomorphic, as Frobenius manifolds, to the spaces of Laurent polynomials $M_{p,q}$ considered by Dubrovin and Zhang ([12]) and Milanov and Tseng ([25]).

4.2 The mirror theorem

In this section we prove the isomorphism between $M_{p,q,r}$ and $QH^*_{\mathrm{orb}}(\mathbb{P}^1_{p,q,r})$, for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, as Frobenius manifolds. Actually, after the work of Milanov and Tseng ([25]), only the cases $M_{2,2,r}$ and the exceptional cases $M_{2,3,3}$, $M_{2,3,4}$, $M_{2,3,5}$ are left to be proven. In order to do that, we are going to show that we can find a point $m \in M_{p,q,r}$ (and hence a dense subset) of our Frobenius manifolds where the linear operator $U : T_m M_{p,q,r} \to T_m M_{p,q,r}$ of multiplication by E_m has pairwise distinct eigenvalues. This ensures (see e.g. [11]) that the Frobenius manifold is semisimple (meaning that the Frobenius algebra is semisimple on a dense subset). Moreover, if we prove that the algebras $T_m M_{p,q,r}$ and $T_m QH^*_{\mathrm{orb}}(\mathbb{P}^1_{p,q,r})$ at m are isomorphic, then the isomorphism of Frobenius structures follows from the identification of e, E and the metric, via the diffeomorphism induced by sending the flat coordinates on $M_{p,q,r}$ to the components of homogeneous basis of cohomology classes in $QH^*_{\mathrm{orb}}(\mathbb{P}^1_{p,q,r})$. In fact (see [11], Lemma 3.3), knowing metric, unity vector field e and Euler vector field E, together with the algebra structure at a point m where $U = (E\bullet_m)$ has pairwise distinct eigenvalues, allows one to reconstruct uniquely the whole Frobenius structure.

After having established the algebra isomorphism, we are going to compute the operator U in the linear basis given by flat coordinates, at the origin 0 of the flat coordinate system. Here eigenvalues can be calculated explicitly but they fail to be distinct. However, in the $M_{2,2,r}$ case, we are able to write down an explicit expression for U along a two-dimensional submanifold of $M_{p,q,r}$ and, by using perturbation theory along this submanifold, we show that degeneracy of the eigenvalues is completely removed by this perturbation. The three exceptional cases can be treated even more easily using the computer to have an explicit expression for U and its eigenvalues at any point of the manifold.

Recall that the orbifold cohomology of $\mathbb{P}^1_{p,q,r}$ as a vector space is just the singular homology of the inertia orbifold of $\mathbb{P}^1_{p,q,r}$, i.e. topologically the disjoint union of a sphere S^2 and $(p+q+r-3)$ isolated points which we denote X_j, Y_k, Z_l with $j = 1,\dots,p-1$, $k = 1,\dots,q-1$, $l = 1,\dots,r-1$. We choose the homogeneous basis of cohomology classes for $QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,2,r})$ given by

$$1 \in H^0(S^2) \quad , \quad p \in H^2(S^2)$$
$$x_1 \in H^0(X_1) \quad , \quad y_1 \in H^0(Y_1)$$
$$z_1 \in H^0(Z_1),\ \dots,\ z_{r-1} \in H^0(Z_{r-1})$$

Let us first compute the quantum algebra structure on $QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,2,r})$ at a point m whose flat coordinates (components on the above basis of cohomology classes) are zero, with the exception of the components a and b along x_1 and y_1, which are left generic.

This structure is a deformation of the ordinary orbifold cup product on $H^*_{\mathrm{orb}}(\mathbb{P}^1_{p,q,r})$, which is given by

$$x_1 \cdot y_1 = x_1 \cdot z_k = y_1 \cdot z_k = 0 \text{ if } k = 1,\dots,r-1$$
$$z_{k_1} \cdot z_{k_2} = x_{k_1+k_2} \text{ if } k_1 + k_2 \le r-1$$
$$2x_1^2 = 2y_1^2 = rz_1^r = p$$

It follows that, as rings,

$$H^*_{orb}(\mathbb{P}^1_{2,2,r}) \simeq \mathbb{C}[x,y,z]/(xy, xz, yz, 2x^2 - 2y^2, 2x^2 - rz^r, 2y^2 - rz^r)$$

if we identify $x_1 = x$, $y_1 = y$, $z_1 = z$.

The main result for determining the algebra structure on $T_m QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,2,r}) = H^*_{\mathrm{orb}}(\mathbb{P}^1_{2,2,r}) \otimes \mathbb{C}[q]$ at m is the following.

Lemma 4.2.1.

$$x_1 \bullet_m z_1 = 2qy_1 + bq$$
$$y_1 \bullet_m z_1 = 2qx_1 + aq$$
$$x_1 \bullet_m y_1 = r(qz_{r-1} + q^3 z_{r-3} + q^5 z_{r-5} + \ldots + q^{2\lfloor\frac{r+1}{2}\rfloor-1} z_{r-(2\lfloor\frac{r+1}{2}\rfloor-1)})$$
$$z_1 \bullet_m z_1 = z_2 + 2q^2$$
$$z_1 \bullet_m z_2 = z_3 + q^2 z_1$$

$$\dots$$

$$z_1 \bullet_m z_{r-2} = z_{r-1} + q^2 z_{r-3}$$

$$z_1 \bullet_m z_{r-1} = p + rq^2 z_{r-2}$$

Proof. Recall firstly that the quantum product agrees with the grading of orbifold cohomology given by

$$\deg z_i = \frac{2i}{r} \qquad \deg x_1 = \deg y_1 = 1 \qquad \deg p = 2$$

$$\deg q = \frac{2}{r} \qquad \deg a = \deg b = 1$$

This gives selection rules on the form of the product. For instance the most general possibility for the product $x_1 \bullet_m z_1$ is

$$x_1 \bullet_m z_1 = 2c_1 x_1 q + 2c_2 y_1 q + rc_3 a z_1 + rc_4 b z_1 + c_5 aq + c_6 bq + r \sum_i c_{7,i} z_i q^{\frac{r+2}{2}-i}$$

where the last sum is only present if r is even and, by the explicit form of the Poincaré pairing and by definition of quantum multiplication (with the usual Gromov-Witten invariants correlator bracket notation $< \dots >_{g,k,d}$, where the indices are respectively genus g, number of marked points k and degree d),

$$c_1 =< x_1, z_1, x_1 >_{0,3,1}= 0$$

$$c_2 =< x_1, z_1, y_1 >_{0,3,1}= 1$$

$$c_3 =< x_1, z_1, z_{r-1}, x_1 >_{0,4,1}= 0$$

$$c_4 =< x_1, z_1, z_{r-1}, y_1 >_{0,4,1}= 0$$

$$c_5 =< x_1, z_1, p, x_1 >_{0,4,1}= 0$$

$$c_6 =< x_1, z_1, p, y_1 >_{0,4,1}= 1$$

$$c_{7,i} =< x_1, z_1, z_i >_{0,4,\frac{r+2}{2}-i}= 0$$

The vanishing of c_1, c_3, c_4, c_5 and the $c_{7,i}$ is proven (see also [25]) by considering a stable map f in the moduli space relevant for the corresponding correlator and the pull back via f of the three line bundles L_x, L_y, L_z which generate the Picard group of $\mathbb{P}^1_{2,2,r}$. The holomorphic Euler characteristics χ of these bundles have to be integer and can be computed explicitly via Riemann-Roch. For instance, for a map f in the moduli space relevant for the correlator c_4, one has

$$\chi(f^* L_x) = 1 + \frac{1}{2} - \frac{1}{2}$$

$$\chi(f^* L_y) = 1 + \frac{1}{2} - \frac{1}{2}$$

$$\chi(f^* L_z) = 1 + \frac{1}{r} - \frac{1}{r} - \frac{r-1}{r}$$

and since the last one is not an integer we deduce that the moduli space is empty and the correlator vanishes. For the same reason c_1, c_3,c_5 and the $c_{7,i}$ vanish, while the correlators $< x_1, z_1, y_1 >_{0,3,1}$ and $< x_1, z_1, p, y_1 >_{0,4,1}$ are clearly 1.

The same technique can be used for all the other relations of the statement but the correlators $< z_1, z_i, z_{i-d+1} >_{0,3,d}$ appearing in $z_1 \bullet_m z_i$. In this case the Euler characteristic only selects d to be even and in order to kill all the terms with $d \neq 2$ one needs to consider a bit more carefully the maps in the relevant moduli space. The key observation here is that any constant component of the map needs extra marked points to be stabilized, while non-constant components need extra marked orbifold points every time they induce a branched covering of S^2 which has local degree not multiple of $2, 2, r$ locally over the orbifold points of stabilizer $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_r$. Plugging this information into Riemann-Hurwitz relation in the same spirit as in the proof of Theorem 3.3.1, one gets the desired vanishing result. $\square$

The above lemma implies

$$T_m QH^*_{\mathrm{orb}}(\mathbb{P}^1_{2,2,r}) \simeq \frac{\mathbb{C}[[q]][x,y,z]}{\begin{pmatrix} xy - rqz^{r-1} + r\sum_{k=1}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^{k-1}(r-2k)\frac{(r-k-1)!}{k!(r-2k)!}q^{2k+1}z^{r-2k-1} \\ xz - 2qy - bq \\ yz - 2qx - aq \end{pmatrix}}$$

where $x_1 = x$, $y_1 = y$ and $z_1 = z$ and where we can set q to any nonzero complex number.

Using the flat coordinates of Lemma 4.1.1 it is easy to see that, at our point m, $c_{r-2k} = (-1)^k r \frac{(r-k-1)!}{k!(r-2k)!}$ for $k = 1, \dots \lfloor \frac{r-1}{2} \rfloor$, which implies that, for q=1 and for every a and b, the one above is precisely the algebra $T_m M_{2,2,r} \simeq \mathbb{C}[x,y,z]/J_{F_m}$ where

$$F_m(x,y,z) = -xyz + x^2 + ax + y^2 + by + z^r + r\sum_{k=1}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \frac{(r-k-1)!}{k!(r-2k)!} q^{2k+1} z^{r-2k}$$

This way we can directly compute the linear operator U of (quantum) multiplication by E at m, whose matrix with respect to the flat coordinates we still denote (with a little abuse of notation) $U = U_0 + V$, where $V(a=0, b=0) = 0$ and

$$V = \begin{pmatrix}
0 & & & & & & & & & & A \\
& -\frac{a^2}{4} & 0 & & & & & & & & \\
& 0 & -\frac{b^2}{4} & & & & & & & & \\
& & & 0 & & & & & & B_{\lfloor \frac{r-1}{2} \rfloor} & \\
& & & \vdots & \ddots & & & & \cdots & & \\
& & & 0 & \dots & 0 & & B_1 & & & \\
& & & 0 & \dots & \dots & 0 & & & & \\
& & & \vdots & \dots & \dots & \vdots & 0 & & & \\
& & & \vdots & \dots & \dots & \vdots & \vdots & \ddots & & \\
& & & 0 & \dots & \dots & 0 & 0 & \dots & 0 & \\
0 & & & & & & & & & & 0
\end{pmatrix}$$

for r even and

$$V = \begin{pmatrix}
0 & & & & & & & & & A \\
 & -\frac{a^2}{4} & 0 & & & & & & & \\
 & 0 & -\frac{b^2}{4} & & & & & & & \\
 & & & 0 & & & & & B_{\lfloor \frac{r-1}{2} \rfloor} & \\
 & & & 0 & \ddots & & & \cdots & & \\
 & & & 0 & \cdots & 0 & B_1 & & & \\
 & & & 0 & \cdots & 0 & 0 & & & \\
 & & & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
 & & & 0 & \cdots & 0 & 0 & \cdots & 0 & \\
0 & & & & & & & & & 0
\end{pmatrix}$$

for r odd, with $A = 2r\frac{r}{2}(a^2+b^2)$, $B_k = \frac{r+2k}{r}k(a^2+b^2)$ for r even and $A = rab$, $B_k = \frac{r+2k-1}{r}(2k-1)ab$ for r odd, and we won't need to know the non-specified entries. While

$$U_0 = \begin{pmatrix}
0 & & & & & & & & 4r \\
 & 2 & 0 & & & & & & \\
 & 0 & 2 & & & & & & \\
 & & & & & & & 2 & \\
 & & & & & & \cdots & & \\
 & & & & & 2 & & & \\
 & & & & \cdots & & & & \\
 & & & 2 & & & & & \\
\frac{1}{r} & & & & & & & & 0
\end{pmatrix}$$

for r even and

$$U_0 = \begin{pmatrix}
0 & & & & & & & & & 4r \\
 & 0 & 2 & & & & & & & \\
 & 2 & 0 & & & & & & & \\
 & & & & & & & & 2 & \\
 & & & & & & & \cdots & & \\
 & & & & & & 2 & & & \\
 & & & & & 2 & & & & \\
 & & & & \cdots & & & & & \\
 & & & 2 & & & & & & \\
\frac{1}{r} & & & & & & & & & 0
\end{pmatrix}$$

for r odd, but here the non-specified entries are zeros.

Now we use perturbation theory to show that U has distinct eigenvalues. In fact let $v_1, \ldots, v_{p+q+r-1}$ be a basis of eigenvectors of U_0, and $\tilde{V}$ the matrix of V in this basis. If $(U_0+\epsilon V)v_i = \lambda(\epsilon)v_i$, then $\lambda(\epsilon) = \lambda(0)+\epsilon\tilde{V}_{ii}+O(\epsilon^2)$. Checking that, for $a \neq b$, the perturbation completely removes the degeneracy of eigenvalues, using the above explicit expressions, is straightforward. This completes the proof of Theorem 4.0.3.

Conclusions

In order to draw the conclusions of our work and propose further developments, we summarize here our results (see also the Introduction for a more detailed review).

In this thesis we basically addressed some Gromov-Witten computations in the target real dimension 2, both in the smooth and orbifold situations. This was done using Symplectic Field Theory as a tool: the target space gets cut to a pair-of-pants decomposition and Symplectic Field Theory techniques are used, first to compute the SFT-potential of the two dimensional cobordisms obtained, then to glue them back and deduce the Gromov-Witten potential of the compact target curve.

This technique is interesting for two reasons, which where actually the motivation for this work. The first reason is effectiveness: Symplectic Field Theory, besides being a very general and beautiful theory of contact invariants, is very powerful for computations in Gromov-Witten theory. Our formulae for target smooth Riemann surfaces are extremely explicit and, for the orbifold case, the insight needed to devise the strucure of the mirror model for polynomial $\mathbb{P}^1$-orbifolds was provided by working out a number of examples where the rational Gromov-Witten potential (potential of the Frobenius manifold) is found in closed form, again by Symplectic Field Theory computations.

The second reason lies in the investigation of the role of integrable systems of Hamiltonian PDEs both in Symplectic Field Theory and Gromov-Witten Theory. In the target two-dimensional case we clarify the role, important also for practical computation, of the dispersionless KdV system and its quantization. The Gromov-Witten Theory of target curves was well-known to involve the Toda integrable system, since the Frobenius structure on quantum cohomology of $\mathbb{P}^1$ is the one giving rise to (extended) Toda equations. This structure, which is semisimple, can be used to compute the full descendant and higher genera Gromov-Witten potential from genus zero and no descendants, thanks to Virasoro constraints. This was basically the road followed in [27], [28] in the smooth situation and [25] in the type A orbifold case. For our approach we used a different technique, related to Symplectic Field Theory of the contact boundaries of the pair-of-pants decomposition of the target spaces, i.e. the circle. In this very simple case Symplectic Field Theory is completely understood to be related to the quantum dispersionless KdV system. In [16] the genus zero potential was already written, while in [32], after an idea of Eliashberg ([15]), we wrote the needed quantization of dispersionless KdV Hamiltonians so that the picture was complete also for the higher genus case. Using Miwa, Jimbo and Date fermionic formalism ([26]), one can promptly handle the mentioned quantum system and get to full descendants all genera Gromov-Witten invariants of any target smooth or orbifold curve, in terms of Hurwitz numbers of branched coverings with assigned branching profile ([32],[33]). Moreover, in this language, it is completely clear how the full descendants potential for the involved cobordisms (the cap and the pair-of-pants) is nothing but a solution to the Schrödinger system of equations associated with the quantum dispersionless KdV system, hence giving a very direct link between the topological theory and integrable systems.

Further developments in this direction are expected to be related with the possibility of using a similar technique in higher target dimension, starting with the four dimensional case. Symplectic Field Theory of S^3 is known to be related to dispersionless Toda system, which is, of course, well known in its classical version (corresponding to genus zero Symplectic Field Theory), but whose quantization is still missing. Once the higher genus SFT-potential for S^3 (and other three-dimensional contact manifolds) is found, possibly in a suitable closed form, one could activate a similar cut-and-paste procedure to attack Gromov-Witten Theory of some class of complex sufaces, admitting some sort of higher dimensional pair-of-pants decomposition. For instance, orbifold Gromov-Witten invariants of weighted projective spaces in any dimension where studied in [24], and their small orbifold quantum cohomology structure was explicitly computed in [7]. It would be interesting to try and approach this computation, starting from weighted projective planes, with our methods to see how integrable hierarchies are involved in the picture and check if one can go beyond small quantum cohomology in explicit terms. Actually, the first steps for this program can already be found in this thesis. The explicit computation of Gromov-Witten potentials for polynomial $\mathbb{P}^1$-orbifolds can be used, together with our orbifold generalization of Bourgeois's theorem ([4]), to get at least the genus zero Symplectic Field Theory Hamiltonians of a class of three-dimensional Seifert fibrations (with up to three singular fibers). Further work needs to be done both about the possibility of extracting contact invariants (basically Contact Homology if not the full Symplectic Field Theory Homology) of the mentioned Seifert fibrations and in the quest for a quantization of these integrable systems (related with the Frobenius manifolds of tri-polynomials we defined). This last task necessarily passes through a better understanding of the classical integrable systems of PDEs arising from the spaces of tri-polynomials. In particular it would be enlightening to find a Lax representation in terms of some suitably generalized shift operators for the D and E cases, similarly to what was done for the A cases by Carlet in his thesis [6].

Less directly related with Symplectic Field Theory, or in a sense complementary to such techniques, is our main result about Mirror Symmetry between $\mathbb{P}^1$-orbifolds, the space of tri-polynomials and extended affine Weyl groups of type A, D, E. While the above methods allow us to compute completely explicitly the Gromov-Witten potential of any assigned specific polynomial $\mathbb{P}^1$-orbifold, we need another kind of approach to tackle all these cases at once, finding the dependence of the Frobenius structure on the three integer parameters encoding the order of the three singular points. Our expression for the quantum cohomology algebra of the general polynomial $\mathbb{P}^1$-orbifold, for instance, relies on our mirror result, whose proof required some techniques in Frobenius manifold theory, namely some of Dubrovin ([10]) and Zhang's ([12]) reconstruction theorems. Namely, after defining a Frobenius structure on the space of tri-polynomials and proving it satisfies the Frobenius manifolds axioms, using the explicit expression of flat coordinates for the invariant metric, we compare the Frobenius algebra, at a given point of our Frobenius manifolds away from the discriminant (where some eigenvalues of the Euler vector field coincide). Once we have shown that these algebras are isomorphic, we use the coincidence of metric and Euler vector field to extend this isomorphism to the whole manifold, basically by analytic continuation. This is essentially different to what Milanov and Tseng did in [25] for the A case. In fact, in the A case, one can do without the assumption to be away from the discriminant, to apply a modified version of the reconstruction theorem at a point where the Euler vector field has an invertible root which generates the algebra. This allows to choose the easiest point in the manifold, the one where all the flat coordinates are equal to zero, even if it is on the discriminant. In the D case, however, the origin of the flat coordinate system is still on the discriminant and, moreover, the root of the Euler vector field which generates the algebra is missing. This left us no choice but to move the reconstruction point away from the origin, which involves computing the quantum cohomology algebra at a non-trivial point. Luckily, in the D case, the coordinates corresponding to orbifold cohomology classes associated to the two $\mathbb{Z}_2$ points are mildly involved in the structure constants of the algebra and the computations can be handled also where these are nonzero. The fact that this point is away from the discriminant then follows by a nice, easy argument in perturbation theory of linear operators.

Here the main direction of further investigation is of course the possibility of generalizing the result to the non-polynomial case, i.e. the computation of Gromov Witten theory of any $\mathbb{P}^1$-orbifold (beyond the already clarified relation with Hurwitz problem of branched coverings). In particular, for the case of only three orbifold points, also after the conjecture of Takahashi ([35]), whose proof should soon be published by the same author, we can already guess that, once more, tri-polynomials of any degree will constitute the candidate for a mirror model. We expect that, in this case, non-polynomiality of the Frobenius manifold can be an obstacle for the computation of the Frobenius algebra at a semisimple point, which is the first step to apply reconstruction theorems in the proof of our mirror theorem. Finally it will also be interesting to see if the Picard-Lefschetz theory approach (see e.g. [34]) can shed some light into the Frobenius structure of these Lefschetz fibrations.

Bibliography

[1] D. Abramovich, *Lectures on Gromov-Witten invariants of orbifolds*, arXiv preprint arXiv:math/0512372.

[2] D. Abramovich, T. Graber and A. Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, arXiv preprint arXiv:math/0603151.

[3] M. Bershadsky, S.Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, Nucl. Phys. **B405** (1993) 279-304.

[4] F. Bourgeois, *A Morse-Bott approach to Contact Homology*, PhD dissertation, Stanford University, 2002.

[5] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, *Compactness results in Symplectic Field Theory*, Geometry and Topology **7**, 2003.

[6] G. Carlet, *The Extended Bigraded Toda hierarchy*, arXiv preprint arXiv:math-ph/0604024.

[7] A. Corti, T. Coates, Y.-P. Lee and H.-H. Tseng, *The quantum orbifold cohomology of weighted projective spaces*, arXiv preprint arXiv:math/0608481.

[8] W. Chen, Y. Ruan, *A New Cohomology Theory of Orbifold* arXiv preprint arXiv:math/0004129v3 [math.AG]

[9] W. Chen, Y. Ruan, *Orbifold Gromov-Witten Theory*, arXiv preprint arXiv:math/0103156v1 [math.AG]

[10] B. Dubrovin, *Geometry of 2D topologica field theories*, Integrable Systems and Quantum Groups, Springer Lecture Notes in Math., vol. 1620 (1996), pp. 120 - 348.

[11] B.Dubrovin, *Painleve' transcendents and topological field theory*, The Painlevproperty: one century later, R.Conte (Ed.), Springer Verlag, 1999, p.287-412.

[12] B. Dubrovin, Y. Zhang, *Extended affine Weyl groups and Frobenius manifolds*, Compositio Mathematica **111**: 167-219, 1998.

[13] B. Dubrovin, Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, preprint arXiv:math.DG/0108160.

[14] Y. Eliashberg, *Symplectic Field Theory and its applications*, Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006.

[15] Y. Eliashberg, *Private communications*, about 2002.

[16] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to Symplectic Field Theory*, GAFA 2000 Visions in Mathematics special volume, part II, 560-673.

[17] T. Eguchi, K. Hori, and C.-S. Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. **B402** (1997), 71-80.

[18] O. Fabert, *A note on integrable systems in Symplectic Field Theory*, preprint.

[19] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173-199.

[20] C. Hertling, *Frobenius manifolds and moduli spaces for singularities*, Cambridge Tracts in Mathematics, 151 (2001).

[21] H. Hofer, *A General Fredholm Theory and Applications*, Current Developments in Mathematics 2004, International Press, Year Published: 2006.

[22] V. G. Kac, *Infinite Dimensional Lie Algebras*, Progress in Mathematics, Birkhäuser 1983.

[23] M. Kontsevich, *Intersection Theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. 147 (1992), no.1, 1-23.

[24] E. Mann, *Cohomologie quantique orbifolde des espaces projectifs poids*, Ph.D. thesis, IRMA, Strasbourg, 2005 and arxiv preprint arXiv:math/0510331v1.

[25] T. E. Milanov, H.-H. Tseng, *The space of Laurent polynomials, $\mathbb{P}^1$-orbifolds, and integrable hierarchies*, preprint arXiv:math/0607012v3 [math.AG]

[26] T. Miwa, M. Jimbo, E. Date, *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras*, Cambridge Tracts in Mathematics, 135. Cambridge University Press, Cambridge 2000.

[27] A. Okounkov, R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, Annals of Math. (2) 163 (2006), 517-560.

[28] A. Okounkov, R. Pandharipande, *Virasoro constraints for target curves*, Invent. Math. 163 (2006), no.1, 47-108.

[29] R. Pandharipande, *The Toda equations and the Gromov-Witten Theory of teh Riemann sphere*, Lett. Math. Phys. **53** (2000), no.1, 59-74.

[30] E.Pervova, C. Petronio, *Realizability and exceptionality of candidate surface branched covers: methods and results*, preprint arXiv:0709.0150.

[31] A. K. Pogrebkov, *Hierarchy of quantum explicitly solvable and integrable models*, preprint arXiv:nlin.SI/0202043v2.

[32] P. Rossi, *Gromov-Witten invariants of target curves via Symplectic Field Theory*, Journal of Geometry and Physics, doi:10.1016/j.geomphys.2008.02.012 and preprint arXiv:0709.2860v1 [math.SG].

[33] P. Rossi, *Gromov-Witten theory of orbicurves, the space of tri-polynomials and Symplectic Field Theory of Seifert fibrations*, preprint arXiv:0808.2626.

[34] P. Seidel, *Fukaya Categories and PicardLefschetz Theory*, Zurich Lectures in Advanced Mathematics, European Math. Soc. Publishing House, 2008.

[35] A. Takahashi, *Weighted projective lines associated to regular systems of weights of dual type*, arXiv preprint arXiv:0711.3907.

[36] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge MA, 1990), 243-310, Lehigh Univ. Bethlehem, PA, 1991.